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Beyond-all-orders effects in multiple-scales asymptotics: travelling-wave solutions to the Kuramoto-Sivashinsky equation

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Abstract. This paper concerns the possible 'shock' patterns that can exist in the solution to a singularly perturbed, third-order nonlinear ordinary differential equation arising as the travelling-wave reduction of the Kuramoto-Sivashinsky equation. In particular, the existence (or otherwise) of oscillatory shocks and multiple shocks made up of combinations of oscillatory and monotonic shocks is examined, using an optimal truncation strategy to track crucial exponentially small terms lying beyond all orders of the (divergent) algebraic expansion. The results provide further understanding of numerical solutions previously obtained by others, as well as giving a methodology which is applicable to much broader classes of differential equations exhibiting multiscale phenomena; they also afford same new insight into the multi-scales technique.

Key words: exponential asymptotics, multiple-scales, Kuramoto-Sivashinsky

1. Introduction

The Kuramoto-Sivashinksy (K-S) equation is relevant to a number of physical applications, including nonlinear chemical kinetics [1], the nonlinear evolution of disturbed flame-fronts [2], [3], instabilities at the interface between viscous fluids [4] and the flow of liquid films down a vertical plane [5]. In the notation introduced in [6], travelling-wave solutions (which are also discussed in a more general setting in [7], [8]) to this equation can be shown, after one integration, to satisfy the third-order nonlinear ordinary differential equation

$$\varepsilon^2 \frac{\mathrm{d}^3 u}{\mathrm{d}z^3} + (1 - 4\varepsilon^2)\frac{\mathrm{d}u}{\mathrm{d}z} = 1 - u^2, \tag{1.1}$$

where z is the travelling-wave coordinate and $0 < \varepsilon < 1/2$; the $-4\varepsilon^2 u'$ term in (1.1) is introduced, in part, to simplify the linearisation analysis to follow shortly (it can be removed by a rescaling and a redefinition of ε). We shall be concerned in this paper with solutions to this equation in the singularly perturbed limit $\varepsilon \to 0$. Before continuing, we remark that same (but not all) of the conclusions of our analysis are already known; a necessarily brief discussion of this is provided later in this section. We thus place some emphasis on the specific methodology that we have adopted to achieve these results, believing that they illustrate that the ideas and techniques in question provide an effective and transparent way of describing delicate asymptotic structures associated with differential equations of significant practical importance. We believe that this work is a further contribution to the development of a widely applicable framework for analysing problems involving asymptotics beyond all orders, particularly for

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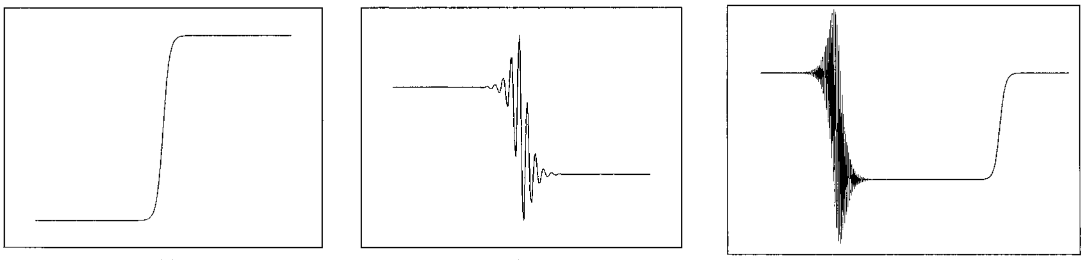

Figure 1. Sections of numerical solutions to (1.1) illustrating (a) a monotonic shock; (b) an oscillatory shock; (c) a solitary wave.

nonlinear systems. To our knowledge, this study involves the first application of the current approach to multiple-scales problems.

In part following previous workers, we define various types of solutions as follows:

Monotonic shock
A monotonic solution connecting the two constant solutions $u = +1$ and $u = -1$ of (1.1) (heteroclinic connection; see Figure 1(a)) .

Oscillatory shock
A global, oscillatory solution connecting $u = 1$ and $u = -1$ (heteroclinic connection; see Figure 1(b)).

Local shock (oscillatory or monotonic)
A solution with one or other of the above behaviours in a localised region, but containing additional exponentially small terms which ultimately causes the solution to diverge from this structure.

Solitary wave
A solution satisfying

$$u \to 1 \text{ as } |z| \to \infty \quad \text{or} \quad u \to -1 \text{ as } |z| \to \infty, z \in \mathbb{R}, \tag{1.2}$$

(homoclinic connection), as illustrated in Figure 1(c).
Detailed numerical studies of such solutions have been performed; see, for example, [9], [10].

If we linearise (1.1) about the solution $u = -1$ in the limit $z \to +\infty$, we obtain the oscillatory behaviour

$$u \sim -1 + M\mathrm{e}^{-z} \sin(\frac{\gamma z}{\varepsilon} + \phi) \quad \text{as} \quad z \to +\infty \tag{1.3}$$

for two arbitrary real constants $\phi \in (0, 2\pi]$ and M, where $\gamma = \sqrt{1 - \varepsilon^2}$. The third solution to the linearised problem, e^{2z}, is discounted because of its exponential growth as $z \to +\infty$. A similar linearisation about the solution $u = 1$ produces

$$u \sim 1 + M\mathrm{e}^{-2z} \quad \text{as} \quad z \to +\infty \tag{1.4}$$

for an arbitrary real constant M; since the other two complementary function grow exponentially (in an oscillatory fashion), they are discounted.

In view of the number of arbitrary constants in (1.3) and (1.4), imposing $u \to -1$ as $z \to +\infty$ is equivalent to imposing a single boundary condition, whereas insisting that $u \to 1$ as $z \to +\infty$ is equivalent to two. The reverse is true when these conditions are imposed as $z \to -\infty$. Such comments have obvious implications regarding the possible existence or otherwise of monotonic and oscillatory shocks. For example, if a monotonic shock exists then it must satisfy

$$\begin{aligned} u &\to -1 \quad \text{as} \quad z \to -\infty, \\ u &\to 1 \quad \text{as} \quad z \to +\infty, \end{aligned} \tag{1.5}$$

whereas for an oscillatory shock we must have

$$\begin{aligned} u &\to 1 \quad \text{as} \quad z \to -\infty, \\ u &\to -1 \quad \text{as} \quad z \to +\infty. \end{aligned} \tag{1.6}$$

In each case, if a solution exists then it is determined only up to translations in z. Given translational invariance and the comments above, (1.5) can be regarded as being equivalent to five boundary conditions and (1.6) as three; however, it should be noted that when restricting attention to odd solutions $u(-z) = -u(z)$, which necessarily satisfy

$$u = \frac{\mathrm{d}^2 u}{\mathrm{d}z^2} = 0 \quad \text{at} \quad z = 0, \tag{1.7}$$

there are actually only four boundary conditions in all when (1.5) and (1.7) hold, whereas for (1.6) and (1.7) there are still three. The oscillatory shock problem therefore appears to be correctly specified, whereas the monotonic shock and solitary wave problems are overspecified by one condition ((1.2) also representing four boundary conditions), but may nonetheless have solutions for particular values of ε.

We mainly concentrate on solutions to (1.1) such that either $u \to 1$ or $u \to -1$ as $z \to +\infty$. It has been shown (for example in [6], [11], [12] in the limit $\varepsilon \to 0$; see also [13], [14] and [15] for another similar application of the Borel resummation technique applied to nonlinear wave problems, as used in [11]) that no monotonic shock solution exists for $\varepsilon > 0$; for $\varepsilon = 0$ the corresponding solution, up to translations in z, is $u = \tanh z$. The non-existence result for small but non-zero ε follows on accounting for the Stokes phenomenon, whereby an initially exponentially small oscillatory term is 'switched on' across a certain line (called a Stokes line) in the complex z-plane. This Stokes line crosses the real z-axis, with significant implications for the asymptotic solutions of (1.1) for physical (*i.e.*, real) values of z (*cf.* [12]). However, strong evidence exists (*e.g.* [9], [10]) to suggest that oscillatory shocks do exist. We shall clarify such results by showing that, although exponentially small terms are also switched on across a Stokes line in this case, there are certain solutions for which the term that would otherwise violate the far-field condition can be suppressed.

The work reported here was completed some time ago (*cf.* [16, Chapters 3 and 4] and the paper [17] has since appeared in which equivalent results were derived by a different technique, and without explicit discussion of the Stokes phenomenon. Our approach provides a more complete description of the behaviour, a point to which we shall return.

As we have already suggested, our analysis is aided by the analytic continuation of the solution into the complex z-plane. Even in the monotonic-shock case we are not aware of any previous analysis of the behaviour of the solution, once a Stokes line has been crossed, other than the solitary-wave solutions described in [17]. We shall provide such an analysis here, showing that the solution typically alternates between local oscillatory and monotonic shocks before ultimately blowing up; if blow-up occurs, it must do so after a local oscillatory shock. We shall also see that each local shock is centred around the intersection of a Stokes line with the real axis, and that this leads to a wide range of possible behaviours. Using asymptotics beyond all orders and matched asymptotic expansions, we are therefore able to explain solutions obtained numerically by Michelson [9] and Hooper and Grimshaw [10], and also to determine asymptotically what type of solution will arise from a particular set of initial conditions. The method we develop here can be extended to cater for other nonlinear differential equations, including the widely-studied model problem in crystal growth considered in, for example, [18], [19].

2. Multiple-scales representation of the oscillatory shock solution

2.1. Early terms of the multiple-scales expansion

We begin by obtaining asymptotic results for the oscillatory shock solutions to (1.1) that have previously been obtained numerically ([9], [10]), thereby extending the results of [16, Chapters 3 and 4], [17] as well as those of Şefik and Ünal [20], who obtained a leading-order solution by different methods. In this section we analyse (odd) solutions to (1.1) which satisfy (1.6) and (1.7). We later clarify the relationship between this set of conditions and (1.6). We obtain the first three terms in the algebraic expansion for u in powers of ε, and subsequently obtain information about the general term in the asymptotic series; from this we deduce the behaviour of the late terms in this series in order to apply optimal truncation techniques.

We notice immediately from (1.3) that there are precisely two lengthscales present in oscillatory shock solutions, namely $z = O(1)$ and $Z \equiv \gamma z/\varepsilon = O(1)$. This strongly motivates us to seek a multiple-scales solution of the form $u(z) \equiv U(z, Z)$; similar analysis of the 'crystal growth' problem has been performed by Dewynne and Robinson [19]. Doing so, we express the K-S equation in the multiple-scales form

$$\gamma^3 U_{ZZZ} + 3\varepsilon\gamma^2 U_{ZZz} + 3\varepsilon^2\gamma U_{Zzz} + \varepsilon^3 U_{zzz} + (1-4\varepsilon^2)(\gamma U_Z + \varepsilon U_z) = \varepsilon(1-U^2). \tag{2.1}$$

We then seek an expansion solution to (2.1) of the form

$$U(z, Z) \sim \sum_{n=0}^{\infty} \varepsilon^n U_n(z, Z) \quad \text{as } \varepsilon \to 0. \tag{2.2}$$

If we now treat z and Z as being independent (as is usual in multiple-scales approaches) then, when (1.3) is satisfied for some M and ϕ (as will be the case for oscillatory shock solutions, for instance; we note that the values of M and ϕ in (1.3) uniquely specify a solution to (1.1) and for the time being we think in terms of an initial value problem from $z = +\infty$), we have the following properties at each order

$$U_n(z + 2\pi\mathrm{i}, Z) = U_n(z, Z), \tag{2.3}$$

$$U_n(z, Z + 2\pi) = U_n(z, Z), \tag{2.4}$$

$$U_n(z + \pi\mathrm{i}, Z + \pi) = U_n(z, Z); \tag{2.5}$$

(2.3) and (2.5) imply (2.4). These periodicity relations are obviously satisfied by the asymptotic limit (1.3), once it has been re-expressed in terms of z and Z. That they are true for all values of z and Z follows because (1.3) represents an initial-value problem and (2.1) is invariant under translations of both variables, changing z to $z+\pi\mathrm{i}$ and Z to $Z+\pi$ (for example, as in (2.5)) leaves the defining Equation (2.1) unchanged and preserves the asymptotic form (1.3), thus mapping the problem into itself, and (2.5) then follows by uniqueness. However, these properties are not satisfied by the exact solution, because of the interdependence of z and Z.

The first three terms in the series (2.2) have been obtained [16, Chapters 3 and 4] by eliminating secularities. For example, writing

$$U_0 = A_0(z)\cos Z + B_0(z)\sin Z + C_0(z), \tag{2.6}$$

which is a consequence of the leading-order balance in (2.1), we obtain the following system of ordinary differential equations for the coefficient functions by eliminating secular terms at $O(\varepsilon)$:

$$\frac{\mathrm{d}A_0}{\mathrm{d}z} = A_0C_0, \tag{2.7}$$

$$\frac{\mathrm{d}B_0}{\mathrm{d}z} = B_0C_0, \tag{2.8}$$

$$\frac{\mathrm{d}C_0}{\mathrm{d}z} = 1 - C_0^2 - \frac{1}{2}(A_0^2 + B_0^2). \tag{2.9}$$

Imposing only the condition in (1.6) as $z \to +\infty$ (other solutions not satisfying this condition are discussed in Appendix A), we see that these differential equations yield

$$\tilde{U}_0 = -2\,\mathrm{sech}\,\tilde{z}\,\sin\tilde{Z} - \tanh\tilde{z}, \tag{2.10}$$

where

$$\tilde{z} = z - \zeta, \quad \tilde{Z} = Z - \mathcal{Z}. \tag{2.11}$$

Here ζ and $\mathcal{Z}$ are arbitrary constants with $\mathcal{Z} \in [0, 2\pi)$, and we write $U_n(z, Z) = \tilde{U}_n(\tilde{z}, \tilde{Z})$. The next two terms in the expansion (2.2) can similarly be derived in the form

$$\tilde{U}_1 = \tfrac{19}{3}\,\mathrm{sech}\,\tilde{z}\,\tanh\tilde{z}\,\cos\tilde{Z} - \tfrac{1}{3}\,\mathrm{sech}^2\tilde{z}\,\sin 2\tilde{Z}, \tag{2.12}$$

$$\begin{aligned}\tilde{U}_2 = {} & -\tfrac{235}{18}\,\mathrm{sech}^2\tilde{z}\,\tanh\tilde{z} + \left(\tfrac{1301}{36}\,\mathrm{sech}\,\tilde{z} - \tfrac{1427}{36}\,\mathrm{sech}^3\tilde{z}\right)\sin\tilde{Z} \\ & + \tfrac{29}{9}\,\mathrm{sech}^2\tilde{z}\,\tanh\tilde{z}\,\cos 2\tilde{Z} - \tfrac{1}{36}\,\mathrm{sech}^3\tilde{z}\,\sin 3\tilde{Z}.\end{aligned} \tag{2.13}$$

Notice that we have omitted complementary function terms in (2.12) and (2.13) since the relevant ones are multiples of $\tilde{U}_{0_{\tilde{z}}}$ and $\tilde{U}_{0_{\tilde{Z}}}$; we can do this without any loss of generality by permitting ζ and $\mathcal{Z}$ to depend on ε. Equations (2.10)–(2.13) evidently satisfy (2.3)–(2.5), and we note that the further symmetry properties

$$\tilde{U}_n(-\tilde{z}, -\tilde{Z}) = -\tilde{U}_n(\tilde{z}, \tilde{Z}), \tag{2.14}$$

$$\tilde{U}_n(\tilde{z}, \pi - \tilde{Z}) = (-1)^n\tilde{U}_n(\tilde{z}, \tilde{Z}), \tag{2.15}$$

also hold at each order when we absorb complementary function terms appropriately into $\zeta(\varepsilon)$ and $\mathcal{Z}(\varepsilon)$ (indeed, it is the identities (2.14–2.15) which specify the quantities $\zeta(\varepsilon)$, $Z(\varepsilon)$ to any order for a given solution to (1.1)).

We now give relationships (which we shall need later) between the constants ζ and $\mathcal{Z}$ and the initial conditions (1.3) which are applicable to other boundary-value problems; without loss of generality (by translating z), we can choose M such that $\zeta = 0$. If we write $M \sim M_0 + \varepsilon M_1 + \varepsilon^2 M_2, \phi \sim \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2$ and $\mathcal{Z} \sim \mathcal{Z}_0 + \varepsilon\mathcal{Z}_1 + \varepsilon^2\mathcal{Z}_2$, imposing (1.3) and matching with (2.10)–(2.13) as $z \to +\infty$ then leads to

$$M_0 = -4,\ \mathcal{Z}_0 = 2\pi - \phi_0,\ M_1 = 0,\ \mathcal{Z}_1 = -\phi_1 - \tfrac{19}{6},\ M_2 = 2\mathcal{Z}_1^2 - 2\phi_1^2 + \tfrac{38}{3}\mathcal{Z}_1 + \tfrac{1301}{18},\ \mathcal{Z}_2 = -\phi_2. \tag{2.16}$$

If we now apply boundary conditions (1.7), we get that $\zeta = 0$ and $\sin\mathcal{Z} = 0$ (to all algebraic orders in ε, *cf.* (2.14), (2.15)), giving (in view of (2.4)) exactly two odd oscillatory shock solutions (corresponding to $\mathcal{Z} = \mathcal{Z}^{(k)}$, $k = 1, 2$, where $\mathcal{Z}^{(1)} = 0$ and $\mathcal{Z}^{(2)} = \pi$). From (2.16), these then correspond to

$$\phi^{(1)}(\varepsilon) = 2\pi - \tfrac{19}{6}\varepsilon + O(\varepsilon^3),\ \ \phi^{(2)}(\varepsilon) = \pi - \tfrac{19}{6}\varepsilon + O(\varepsilon^3), \tag{2.17}$$

respectively, both having

$$M(\varepsilon) = -4 + \tfrac{470}{9}\varepsilon^2 + O(\varepsilon^4). \tag{2.18}$$

These values are in very good agreement with numerical solutions obtained for oscillatory shocks, examples of which are shown in Figure 2.

Writing the two odd oscillatory shock solutions as $U^{(1)}(z, Z)$ and $U^{(2)}(z, Z)$, we have at each order

$$U_n^{(1)}(z, Z) = U_n^{(2)}(z, Z + \pi) = (-1)^n U_n^{(2)}(z, -Z) \tag{2.19}$$

and, in view of (2.15),

$$M(\varepsilon) \sim M(-\varepsilon), \phi^{(2)}(\varepsilon) \sim \pi + \phi^{(1)}(\varepsilon) \sim 2\pi - \phi^{(2)}(-\varepsilon); \tag{2.20}$$

to all algebraic orders. The invariance of (2.1) under the transformations $Z \mapsto -Z$, $\varepsilon \mapsto -\varepsilon$ (which also relies on the fact that z and Z are treated as being independent) plays a key role in the derivation of such results.

It is natural to ask whether (non-antisymmetric) oscillatory shock solutions exist for other values of $\mathcal{Z}$, since our current results suggest that they do for any $\mathcal{Z}$ because (2.10)–(2.13) always satisfy (1.6). We shall see later that this is not the case, but to establish this we shall need to go ‘beyond all orders’ of the algebraic asymptotic expansion (2.2) in order to obtain an exponentially accurate asymptotic solution. To achieve this, we shall truncate the expansion optimally after finding its smallest term $n = N(\varepsilon)$ and then perform an asymptotic analysis on the remainder; as usual, the remainder is then exponentially small (see [21, Chapter 1] for a general discussion of such matters and [12] for further examples). The need to go beyond all orders can readily be seen, as follows: if (2.2) is the asymptotic expansion of a particular solution to (1.1) then

$$U(z, Z) \sim \sum_{n=0}^{\infty} \varepsilon^n U_n(z - \zeta, Z - \mathcal{Z}) \quad \text{as } \varepsilon \to 0 \tag{2.21}$$

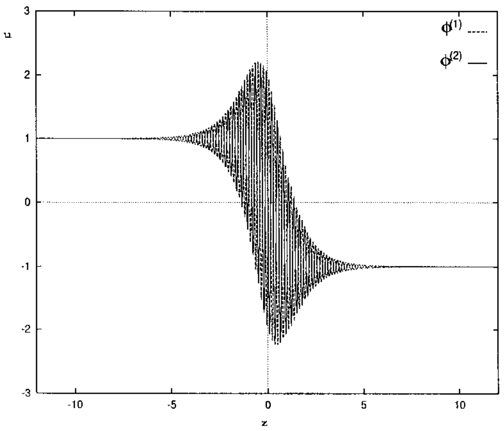

Figure 2. Numerical oscillatory shock solutions to (1.1) subject to (1.3), with $\varepsilon = 0{\cdot}045$, $M = -4 + \frac{470}{9}\varepsilon^2$, $\phi^{(1)} = 6{\cdot}145$, $\phi^{(2)} = 3{\cdot}003$. The choice of M was based on the asymptotic result (2.18) and the values of ϕ were found by iterating on the initial value problem (1.3) until the required behaviour as $z \to -\infty$ was obtained.

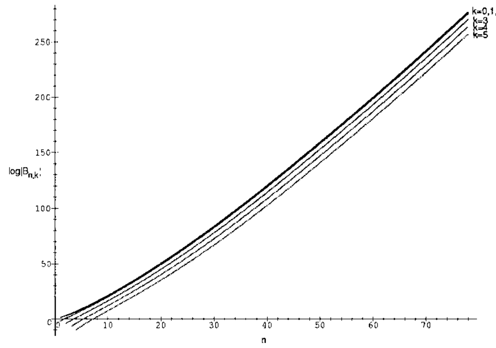

Figure 3. $\log|B_{n,k}|$ against n for $k = 0, 1, \ldots 5$. The curves for $k = 0, 1, 2$ are almost indistinguishable for large n.

gives, for the same functions U_n, the asymptotic expansion of a two-parameter (ζ and $\mathcal{Z}$) family of solutions. When (2.2) is taken to be the asymptotic expansion of an oscillatory shock, the existence of a single such solution implies that the $U_n(z, Z)$ (necessarily satisfying $U_n \to 0$ as $z \to \pm\infty$ for $n \geq 1$) exist to all orders n. It follows that, for any ζ and $\mathcal{Z}$, (2.21) also satisfies $U \to \mp 1$ as $z \to \pm\infty$ to all orders.
Hence, only on going beyond all orders (in our case using optimal truncation) does the non-existence of a one-parameter family of distinct oscillatory shocks emerge asymptotically. We take the parameter to be $\mathcal{Z}$ and, for given ζ, solutions exist only for discrete values of $\mathcal{Z}$; arbitrary translations of z are of course permitted, but we shall not view these as yielding distinct solutions (without loss of generality we again set $\zeta = 0$ by translating z by ζ and replacing $\mathcal{Z}$ by $\mathcal{Z} + \gamma\zeta/\varepsilon$).

2.2. General term in the multiple-scales expansion

2.2.1. *Formulation*

We now consider the form of the general term $U_n(z, Z)$ in (2.2) for the odd oscillatory shock solution with $\zeta = \mathcal{Z} = 0$, the results immediately generalising to the two parameter family (2.21). We can see from (2.10) that the two singularities in z of U_0 which are closest to the real axis are located at $z = \pm\frac{i\pi}{2}$; we shall see later how contributions from these singularities dominate the late terms in (2.2) sufficiently close to the real z-axis. To do so we must first determine the behaviour of each term in (2.2) as z approaches one of these singularities.

Substituting (2.2) in (2.1) and equating terms of $O(\varepsilon^n)$ gives, for $n \geq 2$, the recursive system of equations

$$U_{nZZZ}+U_{nZ} = -3U_{n-1_{ZZz}}-3U_{n-2_{Zzz}}-U_{n-3_{zzz}}-U_{n-1_z}+4U_{n-2_Z}+4U_{n-3_z}+U_{n-2_{ZZZ}}$$
$$+3U_{n-3_{ZZz}}-\sum_{j=0}^{n-1}U_jU_{n-1-j}+\sum_{j=1}^{[\frac{n}{2}]}\frac{(2j-2)!}{2^{2j-1}j!(j-1)!}(U_{n-2j_{ZZZ}}+U_{n-2j_Z})$$
$$+\sum_{j=1}^{[\frac{n}{2}]-1}\frac{(2j-2)!}{2^{2j-1}j!(j-1)!}(-U_{n-2j-2_{ZZZ}}+3U_{n-2j-2_{Zzz}}-4U_{n-2j-2_Z}), \quad (2.22)$$

where $U_n\equiv 0$ for $n<0$, $[\frac{n}{2}]$ is the integer part of $\frac{n}{2}$ and the $\sum_{j=1}^{[\frac{n}{2}]-2}$ sum is taken to be absent for $n=2$ and 3. The form of solution for each n can be shown to be

$$U_n(z,Z)=\sum_{k=-(n+1)}^{n+1}A_{n,k}(z)\mathrm{e}^{k\mathrm{i}Z}. \quad (2.23)$$

The terms in (2.23) with $k=0,\pm 1$ represent the complementary function in the solution to (2.22) and the associated coefficient functions $A_{n,k}(z)$ are determined by the elimination of secular terms in $U_{n+1}(z,Z)$. The other terms, for which $k\neq 0,\pm 1$, are the particular integrals in the solution to (2.22). The behaviour of the functions $A_{n,k}(z)$ in the limit $z\to\frac{\mathrm{i}\pi}{2}$ will provide us with the information which we shall use later in Section 2.2.3. The corresponding results for the other closest singularity, at $z=-\frac{\mathrm{i}\pi}{2}$, can be deduced straight forwardly from symmetry considerations.

2.2.2. *Solution near the singularity* $z=\frac{\mathrm{i}\pi}{2}$

It follows from (2.10), (2.12) and (2.13) that U_n has a singularity of the form $(z-\frac{\mathrm{i}\pi}{2})^{-(n+1)}$ and it can be shown by induction that this behaviour applies for all n. We therefore seek a local solution to (2.22) in the form

$$U_n(z,Z)\sim\frac{\psi_n(Z)}{(z-\frac{\mathrm{i}\pi}{2})^{n+1}} \quad \text{as} \quad z\to\frac{\mathrm{i}\pi}{2}, \quad (2.24)$$

giving (from (2.22)) that

$$\frac{\mathrm{d}^3\psi_n}{\mathrm{d}Z^3}+\frac{\mathrm{d}\psi_n}{\mathrm{d}Z} = 3n\frac{\mathrm{d}^2\psi_{n-1}}{\mathrm{d}Z^2}-3n(n-1)\frac{\mathrm{d}\psi_{n-2}}{\mathrm{d}Z}+n(n-1)(n-2)\psi_{n-3}+n\psi_{n-1}$$
$$-\sum_{j=0}^{n-1}\psi_j\psi_{n-1-j}. \quad (2.25)$$

In view of (2.23) we can write

$$\psi_n(Z)=\sum_{k=-(n+1)}^{n+1}B_{n,k}\mathrm{e}^{k\mathrm{i}Z}, \quad (2.26)$$

the $B_{n,k}$ being as yet undetermined constants. The leading-order solution (2.10) implies

$$B_{0,0}=-1,\ B_{0,1}=1,\ B_{0,-1}=-1, \quad (2.27)$$

consistent with (2.25) for $n=1$, and from (2.12) we also have that

$$B_{1,0} = 0,\ B_{1,1} = -19\mathrm{i}/6,\ B_{1,-1} = -19\mathrm{i}/6,\ B_{1,2} = -\mathrm{i}/6,\ B_{1,-2} = \mathrm{i}/6. \tag{2.28}$$

Substituting (2.26) in (2.25) and equating coefficients of $\mathrm{e}^{k\mathrm{i}Z}$ leads to the difference equation

$$\begin{aligned} -\mathrm{i}k(k^2-1)B_{n,k} = & -n(3k^2-1)B_{n-1,k} - 3n(n-1)\mathrm{i}kB_{n-2,k} + n(n-1)(n-2)B_{n-3,k} \\ & - \sum_{j=0}^{n-1} \sum_{p=p_1}^{p_2} B_{j,p} B_{n-1-j,k-p} \end{aligned} \tag{2.29}$$

for $n \geq 2, k = 0, \pm 1, \ldots, \pm(n+1)$. The summation limits p_1 and p_2 are given by $p_1 = \max\{-(j+1), k-n+j\}$ and $p_2 = \min\{j+1, k+n-j\}$, and we have that $B_{n,k} \equiv 0$ for $n < 0$ and for $|k| > n+1$. For each n, the coefficients $B_{n-1,k}$ for $k = 0, \pm 1$ and $B_{n,k}$ for $k = \pm 2, \pm 3 \ldots \pm (n+1)$ can be obtained from (2.29). This corresponds to the terms in $\exp(k\mathrm{i}Z)$ for $k = 0, \pm 1$ forming the complementary function in (2.22), so those $B_{n-1,k}$ are obtained via secularity conditions; other values of k correspond to the particular integral.

The coefficients $B_{n,k}$ have been calculated iteratively from (2.29) up to a large value of n using exact arithmetic in Maple. The following observations can be made:

OB(1). $B_{n,-k} = (-1)^{n+k} B_{n,k}$.
OB(2). $B_{n,k}/(-\mathrm{i})^n$ is real and positive for $k > 0$.
OB(3). $B_{n,0} = 0$ for n odd.
OB(4). $B_{n,0}/\mathrm{i}^n$ is real and negative for n even.
OB(5). As n becomes large, those $B_{n,k}$ with $|k| > 2$ are negligible in comparison with those for $|k| \leq 2$. Those with $|k| \leq 2$ are all comparable.

OB(3) is a trivial consequence of OB(1), which in turn can easily be proved by induction; the first parts of OB(2) and OB(4) are immediate from (2.29) and the second parts can be verified *a posteriori* (see later); OB(5) is illustrated in Figure 3 and is justified in Appendix B.

Using these observations in the limit $n \to \infty$, we can reduce the system (2.29) of $2n+3$ difference equations to a system of three equations, namely those corresponding to $k = 0, 1, 2$. We shall now derive expressions for $B_{n,0}$, $B_{n,1}$ and $B_{n,2}$ as $n \to \infty$, which suffice to determine the asymptotic form for the general term U_n as $z \to \frac{\mathrm{i}\pi}{2}$, $n \to \infty$, given (2.24), (2.26), OB(1) and OB(5).

A consequence of (2.29) is that the terms which are linear in the B's are each of the order $n^{p+1} B_{n-p,k}$ as $n \to \infty$ for some p. Following a similar approach to that given in Chapman *et al.* [12] leads us, in order to balance these linear terms, to write the $B_{n,k}$ in the form

$$\begin{aligned} B_{n,0} &= -\mathrm{i}^n \Gamma(n+\alpha_0+1)\lambda_{n,0}, \quad n \text{ even}, \\ &= 0, \quad n \text{ odd}, \end{aligned} \tag{2.30}$$

$$B_{n,k} = (-\mathrm{i})^n \Gamma(n+\alpha_k+1)\lambda_{n,k}, \quad k \geq 1, \tag{2.31}$$

where the α_k are independent of n and the $\lambda_{n,k}$ are positive and $O(1)$ as $n \to \infty$. We now seek $\Lambda_k = \lim_{n\to\infty} \lambda_{n,k}$ for $k = 0, 1, 2$, introducing the expansions $\lambda_{n,k} \sim \Lambda_k + \frac{1}{n}(l_k + (-1)^n m_k)$ as $n \to \infty$; we can set $m_0 = 0$ (see (2.30)).

In view of OB(5), we anticipate that we may write $\alpha_k = \alpha$ for $k = 0, 1, 2$ (*cf.* Appendix B, where it is also shown that $\alpha_k < \alpha$ for $k > 2$, and Figure 3). Substituting $k = 0$ in (2.29) gives, we have neglecting the $B_{n,k}$ with $|k| > 2$ because of OB(5),

$$nB_{n-1,0} + n(n-1)(n-2)B_{n-3,0} \sim 2(B_{0,-1}B_{n-1,1} + B_{0,0}B_{n-1,0} + B_{0,1}B_{n-1,-1}) \tag{2.32}$$

for n odd; for n even, (2.29) is identically zero for $k = 0$. Substitution of the initial values from (2.27) and using (2.30), (2.31) and OB(1), (2.32) implies at leading order that

$$(-2\alpha - 2)\Lambda_0 + 4\Lambda_1 = 0 \quad \text{as} \quad n \to \infty. \tag{2.33}$$

The analogue of (2.32) for $k = 1$ is

$$\begin{aligned} -2nB_{n-1,1} &+n(n-1)(n-2)B_{n-3,1} - 3\mathrm{i}n(n-1)B_{n-2,1} \sim 2(B_{0,-1}B_{n-1,2} \\ &+B_{0,0}B_{n-1,1} + B_{0,1}B_{n-1,0}). \end{aligned} \tag{2.34}$$

When (2.30) and (2.31) are used, (2.34) simplifies at leading order to

$$(2-\alpha)\Lambda_1 + 2\Lambda_2 + \Lambda_0 + (-1)^{n+1}\Lambda_0 + 6(-1)^n m_1 = 0. \tag{2.35}$$

Finally, for $k = 2$ the leading-order approximation to the difference equation (2.29) reduces

$$\begin{aligned} -6\mathrm{i}B_{n,2} &+11nB_{n-1,2} - n(n-1)(n-2)B_{n-3,2} + 6\mathrm{i}n(n-1)B_{n-2,2} \\ &\sim 2(B_{0,0}B_{n-1,2} + B_{0,1}B_{n-1,1}), \end{aligned} \tag{2.36}$$

and a similar analysis as $n \to \infty$ leads to

$$(\alpha+1)\Lambda_2 - \Lambda_1 + 24(-1)^n m_2 = 0. \tag{2.37}$$

By considering separately the odd and even (in n) versions of (2.33), (2.35) and (2.37), we obtain $m_1 = \Lambda_0/6$, $m_2 = 0$ and, more importantly,

$$(-2\alpha - 2)\Lambda_0 + 4\Lambda_1 = 0, \tag{2.38}$$

$$(2-\alpha)\Lambda_1 + 2\Lambda_2 + \Lambda_0 = 0, \tag{2.39}$$

$$(\alpha+1)\Lambda_2 - \Lambda_1 = 0. \tag{2.40}$$

The three non-trivial possible solutions of these are

$$\alpha = 3,\ \Lambda_1 = 2\Lambda_0,\ \Lambda_2 = \tfrac{1}{2}\Lambda_0, \tag{2.41}$$

$$\alpha = -1,\ \Lambda_2 = -\tfrac{1}{2}\Lambda_0,\ \Lambda_1 = 0, \tag{2.42}$$

$$\alpha = -2,\ \Lambda_1 = -\tfrac{1}{2}\Lambda_0,\ \Lambda_2 = \tfrac{1}{2}\Lambda_0, \tag{2.43}$$

and we note that the values of Λ_0 will differ in these three separate cases. From (2.30)–(2.31) the solution with the largest α dominates for large n, so (2.42) and (2.43) represent negligible contributions to the $B_{n,k}$ compared to (2.41); this is consistent with (2.30) which implies that the Λ_k must all be positive for the dominant large n expression. The relationships between the Λ_k in (2.41) are in very good agreement with the exact results obtained via Maple (see Figure 4), from which Λ_k can be estimated as

$$\Lambda_0 \approx 0{\cdot}173, \quad \Lambda_1 \approx 0{\cdot}346, \quad \Lambda_2 \approx 0{\cdot}086; \tag{2.44}$$

the value of Λ_1, say, can be found only by such an approach of iterating (2.29) all the way up to a suitably large value of n – it cannot be determined purely from the asymptotic analysis of the $n \to \infty$ limit. Once Λ_1 is known, the values of Λ_2 and Λ_0 are given by (2.41). These values of Λ_k can be shown to be consistent with the value $C = 0{\cdot}26$ obtained by Yang [17]. Combining the results so far, we have that

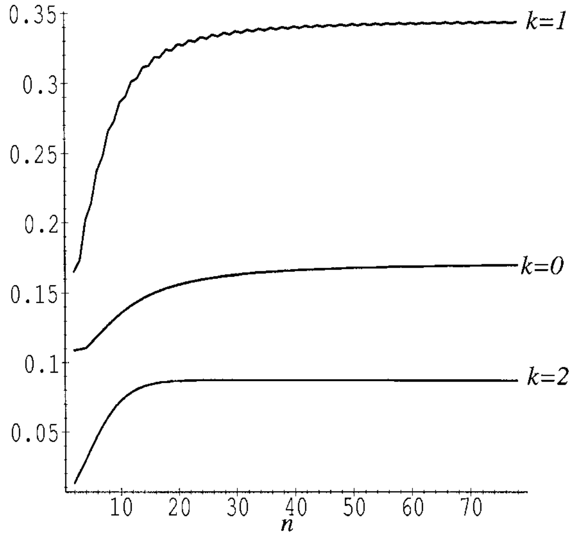

Figure 4. $\lambda_{n,k}$ against n for $k = 0, 1, 2$

$$U_n \sim \frac{(n+3)!\Lambda_1(-\mathrm{i})^n\{\mathrm{e}^{\mathrm{i}Z} + (-1)^{n+1}\mathrm{e}^{-\mathrm{i}Z} - \frac{1}{4}(1+(-1)^n) + \frac{1}{4}(\mathrm{e}^{2\mathrm{i}Z} + (-1)^n\mathrm{e}^{-2\mathrm{i}Z})\}}{(z - \frac{\mathrm{i}\pi}{2})^{n+1}}, \quad (2.45)$$

as $z \to \frac{\mathrm{i}\pi}{2}$, $n \to \infty$. The corresponding result at the other nearest singularity $z = -\frac{\mathrm{i}\pi}{2}$ is

$$U_n \sim \frac{(n+3)!\Lambda_1\mathrm{i}^n\{\mathrm{e}^{-\mathrm{i}Z} + (-1)^{n+1}\mathrm{e}^{\mathrm{i}Z} - \frac{1}{4}(1+(-1)^n) + \frac{1}{4}(\mathrm{e}^{-2\mathrm{i}Z} + (-1)^n\mathrm{e}^{2\mathrm{i}Z})\}}{(z + \frac{\mathrm{i}\pi}{2})^{n+1}}, \quad (2.46)$$

as $z \to -\frac{\mathrm{i}\pi}{2}$, $n \to \infty$.

We end this section by noting that the validity of the latter parts of the observations OB(2) and OB(4) for large n is implicit in our calculation of the real $O(1)$ quantities Λ_0, Λ_1 and Λ_2. For $n = O(1)$, they are apparent from inspection of the exact values obtained for the $B_{n,k}$.

2.2.3. *Extension to the rest of the complex z-plane*

In order to generalise the representation (2.45) for U_n to other values of z, particularly for real z, we follow an approach similar to that of Chapman *et al.* [12]. In view of the results of Section 2.2.2, we seek a solution of the form

$$U_n(z, Z) \sim \frac{(-\mathrm{i}^n(n+\alpha)!}{(z - \frac{\mathrm{i}\pi}{2})^{n+\alpha+1}} \sum_{k=-3}^{3} f_{n,k}(z)\mathrm{e}^{k\mathrm{i}Z} \quad \text{as } n \to \infty, \quad (2.47)$$

where the $(z - \frac{\mathrm{i}\pi}{2})^{-(n+\alpha+1)}$ pre-factor is guided by (2.24) and we remark that including the α in the exponent (rather than including that factor within the $f_{n,k}$) simplifies subsequent calculations. It follows from (2.24) that $f_{n,k} = O((z - \frac{\mathrm{i}\pi}{2})^\alpha)$ as $z \to \frac{\mathrm{i}\pi}{2}$ and, from (2.45), that $\alpha = 3$; however, we continue to perform a general analysis since this will enable us to interpret the solutions (2.42)–(2.43). We shall also confirm our earlier assertion that the terms for which $|k| \geq 3$ are negligible in comparison with those for which $|k| \leq 2$ (nevertheless, the terms with $k = \pm 3$ must be included in (2.47), since we shall require information from these correction terms). It is again necessary to split the solution into odd and even terms in n, and we write terms

$$f_{n,k}(z) = F_k^{(0)}(z) + (-1)^n G_k^{(0)}(z) + \frac{1}{n}\left(F_k^{(1)}(z) + (-1)^n G_k^{(1)}(z)\right) + \ldots \text{ as } n \to \infty. \tag{2.48}$$

It follows from (2.15) that

$$f_{n,-k}(z) = (-1)^{n+k} F_{n,k}(z), \tag{2.49}$$

and hence that

$$G_k^{(j)}(z) = (-1)^k F_{-k}^{(j)}(z). \tag{2.50}$$

Substituting (2.47) and (2.48) in (2.22) and retaining the first two orders yields, after significant simplification,

$$\begin{aligned}
&\sum_{k=-3}^{3}(k-k^3)\left((n+3)(F_k^{(0)}+(-1)^n G_k^{(0)}) + F_k^{(1)} + (-1)^n G_k^{(1)}\right)\mathrm{e}^{k\mathrm{i}Z} = \\
&\sum_{k=-3}^{3}\Big\{(-3k^2+3k)\left((n+3)F_k^{(0)} + F_k^{(1)}\right) + (-1)^n(3k^2+3k)\left((n+3)G_k^{(0)} + G_k^{(1)}\right) \\
&+(3k^2-6k+2)(z-\tfrac{\mathrm{i}\pi}{2})F_k^{(0)\prime} + (-1)^n(-3k^2-6k-2)(z-\tfrac{\mathrm{i}\pi}{2})G_k^{(0)\prime} \\
&-2U_0(z,Z)(z-\tfrac{\mathrm{i}\pi}{2})(F_k^{(0)} + (-1)^{n-1}G_k^{(0)})\Big\}\mathrm{e}^{k\mathrm{i}Z},
\end{aligned} \tag{2.51}$$

where $U_0(z, Z)$ is given by (2.10).

Equating coefficients of $\exp(k\mathrm{i}Z)$ at $O(n)$ in (2.51) gives

$$F_{-3}^{(0)} = G_{-3}^{(0)} = F_{-2}^{(0)} = F_{-1}^{(0)} = G_1^{(0)} = G_2^{(0)} = F_3^{(0)} = G_3^{(0)} = 0, \tag{2.52}$$

and, at $O(1)$, the system of ordinary differential equations

$$F_0^{(0)\prime} = -\tanh z\, F_0^{(0)} - \mathrm{i}\,\mathrm{sech}\, z\, F_1^{(0)} \tag{2.53}$$

$$F_1^{(0)\prime} = -2\mathrm{i}\,\mathrm{sech}\, z\, F_0^{(0)} + 2\tanh z\, F_1^{(0)} + 2\mathrm{i}\,\mathrm{sech}\, z\, F_2^{(0)} \tag{2.54}$$

$$F_2^{(0)\prime} = \mathrm{i}\,\mathrm{sech}\, z\, F_1^{(0)} - \tanh z\, F_2^{(0)} \tag{2.55}$$

results. We also obtain equations for the $G_k^{(0)}$ which are consistent with (2.50) and, using (2.52), the following information on the correction terms:

$$\begin{aligned}
&F_{-3}^{(1)} = F_{-2}^{(1)} = G_2^{(1)} = G_3^{(1)} = 0,\ \ G_{-3}^{(1)} = -\tfrac{\mathrm{i}}{3}(z-\tfrac{\mathrm{i}\pi}{2})\mathrm{sech}\, z\, G_{-2}^{(0)}, \\
&F_{-1}^{(1)} = -\tfrac{\mathrm{i}}{3}(z-\tfrac{\mathrm{i}\pi}{2})\mathrm{sech}\, z\, F_0^{(0)},\ \ G_1^{(1)} = \tfrac{\mathrm{i}}{3}(z-\tfrac{\mathrm{i}\pi}{2})\mathrm{sech}\, z\, G_0^{(0)},\ \ F_3^{(1)} = \tfrac{\mathrm{i}}{3}(z-\tfrac{\mathrm{i}\pi}{2})\mathrm{sech}\, z\, F_2^{(0)}.
\end{aligned} \tag{2.56}$$

The general solution to (2.53)–(2.55) takes the form

$$F_0^{(0)} = \mathrm{i}K_1\left(3\,\mathrm{sech}\, z - \cosh z - 3(z-\frac{\mathrm{i}\pi}{2})\mathrm{sech}\, z\tanh z\right) + K_2\mathrm{sech}\, z - \mathrm{i}K_3\,\mathrm{sech}\, z\tanh z, \tag{2.57}$$

$$F_1^{(0)} = K_1\left(2\sinh z\cosh z + 3\tanh z + 3(z-\frac{\mathrm{i}\pi}{2})\,\mathrm{sech}^2 z\right) + K_3\,\mathrm{sech}^2 z, \tag{2.58}$$

$$F_2^{(0)} = -\mathrm{i}K_1\left(3\,\mathrm{sech}\,z - \cosh\,z - 3(z - \frac{\mathrm{i}\pi}{2})\mathrm{sech}\,z\,\tanh\,z\right) + K_2\,\mathrm{sech}\,z + \mathrm{i}K_3\,\mathrm{sech}\,z\,\tanh\,z, \tag{2.59}$$

where K_1, K_2 and K_3 are arbitrary constants. As $z \to \mathrm{i}\pi/2$, the K_1 terms are $O((z - \frac{\mathrm{i}\pi}{2})^3)$, the K_2 terms are $O((z - \frac{\mathrm{i}\pi}{2})^{-1})$ and the K_3 terms $O((z - \frac{\mathrm{i}\pi}{2})^{-2})$. Thus to match with (2.41), (2.45) we require

$$\alpha = 3,\, K_1 = -\tfrac{5}{4}\Lambda_0,\;\; K_2 = K_3 = 0 \tag{2.60}$$

and this term dominates the large-n behaviour. However, the other two terms will also in general be present, but will be algebraically smaller in n (see (2.47)). These are

$$\alpha = -1,\;\; K_2 = -\tfrac{\mathrm{i}}{2}\Lambda_0,\;\; K_3 = 0, \tag{2.61}$$

$$\alpha = -2,\;\; K_3 = \tfrac{1}{2}\Lambda_0, \tag{2.62}$$

where the Λ_0 are those arising in (2.42)–(2.43), with distinct values for each of these roots.

To summarise, the dominant behaviour for the U_n as $n \to \infty$ is given by setting $\alpha = 3$ in (2.47) with the $f_{n,k}$'s given by (2.48), (2.49), (2.52) and

$$F_0^{(0)} = -\frac{5\mathrm{i}}{4}\Lambda_0\left(3\,\mathrm{sech}\,z - \cosh\,z - 3\left(z - \frac{\mathrm{i}\pi}{2}\right)\,\mathrm{sech}\,z\,\tanh\,z\right) = G_0^{(0)} = -F_2^{(0)} = -G_{-2}^{(0)}, \tag{2.63}$$

$$F_1^{(0)} = -\frac{5}{4}\Lambda_0\left(2\,\sinh\,z\,\cosh\,z + 3\,\tanh\,z + 3\left(z - \frac{\mathrm{i}\pi}{2}\right)\,\mathrm{sech}^2 z\right) = -G_{-1}^{(0)}. \tag{2.64}$$

The constant Λ_0 is given numerically by (2.44) as 0·173. The correction terms $F_k^{(1)}$ and $G_k^{(1)}$ are not required for our subsequent leading-order calculations, though their inclusion here was necessary in order to derive Equations (2.53)–(2.55) for the $F_k^{(0)}$ and $G_k^{(0)}$. In particular, we have confirmed that, since $F_{\pm 3,0} = G_{\pm 3,0} = 0$, the $f_{n,\pm 3}$ are of the same order as $f_{n,k}/n$ with $|k| \le 2$, in the limit $n \to \infty$.

We can now use symmetry arguments to deduce that the contribution to U_n from the singularity at $z = -\frac{\mathrm{i}\pi}{2}$ is given by

$$U_n \sim \frac{\mathrm{i}^n (n+3)!}{(z + \frac{\mathrm{i}\pi}{2})^{n+4}} \sum_{k=-2}^{k=2} \hat{f}_{n,k}(z)\mathrm{e}^{k\mathrm{i}Z}, \tag{2.65}$$

where we can infer from (2.63), (2.64) that as $n \to \infty$

$$\begin{aligned}
\hat{f}_{n,0} &\sim \frac{5\mathrm{i}}{4}\Lambda_0(1 + (-1)^n)\left(3\,\mathrm{sech}\,z - \cosh\,z - 3\left(z + \frac{\mathrm{i}\pi}{2}\right)\,\mathrm{sech}\,z\,\tanh\,z\right), \\
\hat{f}_{n,-1} &= (-1)^{n+1}\hat{f}_{n,1} \sim -\frac{5\mathrm{i}}{4}\Lambda_0\left(2\,\sinh\,z\,\cosh\,z + 3\,\tanh\,z + 3\left(z + \frac{\mathrm{i}\pi}{2}\right)\mathrm{sech}^2 z\right), \\
\hat{f}_{n,-2} &= (-1)^n \hat{f}_{n,2} \sim -\frac{5\mathrm{i}}{4}\Lambda_0\left(3\,\mathrm{sech}\,z - \cosh\,z - 3\left(z + \frac{\mathrm{i}\pi}{2}\right)\,\mathrm{sech}\,z\,\tanh\,z\right).
\end{aligned} \tag{2.66}$$

The full expression U_n as $n \to \infty$ with z real is given by the sum of (2.47) and (2.65). There are similar contributions to U_n from the singularities at $z = \sigma_m$, where $\sigma_m = (2m-1)\mathrm{i}\pi/2$ for integer m; since these contain the factor $(z - \sigma_m)^{-n}$ as $n \to \infty$, the others are exponentially smaller near the real z-axis than those already considered (namely $m = 0, 1$) and can be discounted.

3. Optimal truncation

3.1. Introduction

We are concerned with the initial-value problem with data prescribed as $z \to +\infty$, in which (1.1) is subject to (1.3) with specified values of M and ϕ. We are now in a position to truncate the expansion (2.21) optimally, the U_n being those whose properties we have discussed in Section 2.2 and where we take $\zeta = 0$. By comparing successive terms of the asymptotic series (2.2) using (2.47), we find that the least term occurs near $n = N$, where $|\varepsilon N/(z - \frac{\mathrm{i}\pi}{2})| \sim 1$. Thus if $z - \frac{\mathrm{i}\pi}{2} = r\mathrm{e}^{\mathrm{i}\omega}(r, \omega \in \mathbb{R})$ then

$$N = \frac{r}{\varepsilon} + \nu \tag{3.1}$$

where $\nu \in [0, 1)$ is included to ensure that N is an integer. We shall be interested in the behaviour of U close to $z = 0$, so we fix $r = \frac{\pi}{2}$ and vary ω about $-\frac{\pi}{2}$ (which is the Stokes line); the optimal truncation point of (2.65) at $z = 0$ is the same as that of (2.47). Henceforth we therefore take

$$N = \frac{\pi}{2\varepsilon} + \nu. \tag{3.2}$$

We now write the exact expression

$$U = \sum_{n=0}^{U} \varepsilon^n U_n(z, \tilde{Z}) + R_N(z, \tilde{Z}), \tag{3.3}$$

where R_N, the remainder after optimal truncation, is exponentially small. Substituting (3.3) in (1.1) yields, as $\varepsilon \to 0$,

$$\begin{aligned} R_{N_{\tilde{Z}\tilde{Z}\tilde{Z}}} + R_{N_{\tilde{Z}}} + \varepsilon\left(3R_{N_{\tilde{Z}\tilde{Z}z}} + R_{N_z} + 2U_0 R_N\right) &\sim \varepsilon^{N+1}(U_{N+1_{\tilde{Z}\tilde{Z}\tilde{Z}}} + U_{N+1_{\tilde{Z}}} \\ &+\varepsilon^{N+2}(-3U_{N_{\tilde{Z}zz}} - U_{N-1_{zzz}}) - \varepsilon^{N+3} U_{N_{zzz}} \end{aligned} \tag{3.4}$$

retaining only those terms which will contribute at leading order within the Stokes layer (as always, correction terms such as R_N are necessarily determined asymptotically by a linear problem, which aids significantly the study of beyond-all-orders effects).

We intend to show that certain terms in the remainder are 'switched on' across a Stokes line, which in this instance intersects the real z-axis at the origin. To do this, we first calculate the remainder within the Stokes layer (*i.e.*, the narrow region in which the Stokes smoothing occurs) and find that these terms are switched on via an error function smoothing, as is typical in both linear [22] and nonlinear [12] problems. We then proceed to solve the differential equation for the remainder outside the Stokes layer, leading to different multiples of solutions to the homogeneous equation occurring on different sides of the Stokes layer. Thus, as usual, these apparent discontinuities in the remainder are smoothed across the Stokes layer.

3.2. Solution to the remainder equation within the Stokes layer

For the calculation pertaining to monotonic shock solutions to the K-S equation ([12], [16, Chapter 4]) the error function smoothing of the Stokes discontinuity occurs across a Stokes layer of width $O(\varepsilon^{\frac{1}{2}})$. We shall now show that a similar situation arises for the oscillatory shock solution being studied here, so we introduce $\hat{z} = \varepsilon^{-\frac{1}{2}} z$ and define $W(\hat{z}, \tilde{Z}) \equiv R_N(z, \tilde{Z})$. The leading-order balance from (3.4) then reduces as $\varepsilon \to 0$, $N \to \infty$ to

$$\begin{aligned} W_{\tilde{Z}\tilde{Z}\tilde{Z}} + \ W_{\tilde{Z}} + \varepsilon^{\frac{1}{2}}(3W_{\tilde{Z}\tilde{Z}\hat{z}} + W_{\hat{z}}) \sim \\ \varepsilon^{N+1}(U_{N+1_{\tilde{Z}\tilde{Z}\tilde{Z}}} + U_{N+1_{\tilde{Z}}}) + \varepsilon^{N+2}\left(-3U_{N_{\tilde{Z}zz}} - U_{N-1_{zzz}}\right) - \varepsilon^{N+3} U_{N_{zzz}}. \end{aligned} \tag{3.5}$$

We denote by rhs^+ the contribution to the right-hand side of (3.5) arising from the singularity at $z = \frac{\mathrm{i}\pi}{2}$, which is given by (2.47). We can expand (3.5), using (2.47), (3.2) and Stirling's formula, to give

$$rhs^+ \sim 4\mathrm{i}\Lambda_0 \varepsilon^{-\frac{7}{2}} \mathrm{e}^{-\frac{\pi}{2\varepsilon}} \mathrm{e}^{-\mathrm{i}z/\varepsilon} \mathrm{e}^{-z^2/(\pi\varepsilon)} \sum_{k=-2}^{2} \left((k - k^3) f_{N+1,k}(z) + (3k-1) f_{N,k}(z) - F_{N-1,k}(z)\right) \mathrm{e}^{k\mathrm{i}\tilde{Z}}, \tag{3.6}$$

in which the $\exp(-\mathrm{i}z/\varepsilon - z^2/(\pi\varepsilon))$ term arises from the small z expansion of the $(z - \frac{\mathrm{i}\pi}{2})^{-\pi/(2\varepsilon)}$ factor in U_N. Noting that $\exp(-\mathrm{i}z/\varepsilon) \sim \exp(-\mathrm{i}(\tilde{Z} + \mathcal{Z}))$, we may simplify this to

$$rhs^+ \sim 4\mathrm{i}\Lambda_0 \varepsilon^{-\frac{7}{2}} \mathrm{e}^{-\frac{\pi}{2\varepsilon}} \mathrm{e}^{-\mathrm{i}\mathcal{Z}} \mathrm{e}^{-\hat{z}^2/\pi} \sum_{k=-2}^{2} c_{N,k-1} \mathrm{e}^{(k-1)\mathrm{i}\tilde{Z}} \tag{3.7}$$

as $\varepsilon \to 0$ with $\hat{z} = O(1)$, where from (2.63), (2.64) we can calculate that

$$c_{N,-3} = (-1)^{N+1} 15\mathrm{i},\ c_{N,-2} = (-1)^N \frac{45\mathrm{i}\pi}{16},\ c_{N,-1} = \frac{5\mathrm{i}}{2},\ c_{N,0} = \frac{15\mathrm{i}\pi}{16},\ c_{N,1} = -\frac{5\mathrm{i}}{2},\ c_{N,2} = c_{N,3} = 0. \tag{3.8}$$

The Stokes layer scaling $\hat{z} = \varepsilon^{-\frac{1}{2}} z$ is motivated by the $\mathrm{e}^{-z^2/(\pi\varepsilon)}$ term in (3.6).

Again using (3.2), we similarly obtain from (2.65) that the contribution from the singularity at $z = -\frac{\mathrm{i}\pi}{2}$ is, in an obvious notation,

$$rhs^- \sim 4\mathrm{i}\Lambda_0 \varepsilon^{-\frac{7}{2}} \mathrm{e}^{-\frac{\pi}{2\varepsilon}} \mathrm{e}^{\mathrm{i}\mathcal{Z}} \mathrm{e}^{-\hat{z}^2/\pi} \sum_{k=-2}^{2} c_{N,-k-1} \mathrm{e}^{(k+1)\mathrm{i}\tilde{Z}}, \tag{3.9}$$

as $\varepsilon \to 0$ with $z = O(1)$. We can now formulate the leading-order remainder equation within the Stokes layer.

Adding the contributions from the two singularities, we obtain

$$W_{\tilde{Z}\tilde{Z}\tilde{Z}} + W_{\tilde{Z}} + \varepsilon^{\frac{1}{2}}(3W_{\tilde{Z}\tilde{Z}\hat{z}} + W_{\hat{z}}) \sim 4\mathrm{i}\Lambda_0 \varepsilon^{-\frac{7}{2}} \mathrm{e}^{-\frac{\pi}{2\varepsilon}} \mathrm{e}^{-\hat{z}^2/\pi} \sum_{k=-3}^{3} \left(c_{N,k} \mathrm{e}^{-\mathrm{i}\mathcal{Z}} + c_{N,-k} \mathrm{e}^{\mathrm{i}\mathcal{Z}}\right) \mathrm{e}^{k\mathrm{i}\tilde{Z}}. \tag{3.10}$$

This motivates us to seek a solution to (3.10) of the form

$$W(\hat{z}, \tilde{Z}) \sim \varepsilon^{-4} \mathrm{e}^{-\frac{\pi}{2\varepsilon}} \sum_{k=-1}^{1} q_{N,k}(\hat{z}) \mathrm{e}^{k\mathrm{i}\tilde{Z}}, \tag{3.11}$$

valid for $\hat{z} = 0(1)$, the terms for $k = \pm 2, \pm 3$ in (3.10) being a factor $\varepsilon^{\frac{1}{2}}$ smaller in magnitude; moreover, such terms in W tend to zero as $\hat{z} \to \pm\infty$ and thus do not switch on anything as the Stokes layer is crossed. The terms in (3.11) are governed by

$$\sum_{k=-1}^{1}(1-3k^2)\frac{\mathrm{d}q_{N,k}}{\mathrm{d}\hat{z}}\mathrm{e}^{k\mathrm{i}\tilde{Z}} = 4\mathrm{i}\Lambda_0\mathrm{e}^{-\hat{z}^2/\pi}\left(\frac{5\mathrm{i}}{2}(\mathrm{e}^{-\mathrm{i}\mathcal{Z}}-\mathrm{e}^{\mathrm{i}\mathcal{Z}})(\mathrm{e}^{-\mathrm{i}\tilde{Z}}-\mathrm{e}^{\mathrm{i}\tilde{Z}})+\frac{15\mathrm{i}\pi}{16}(\mathrm{e}^{-\mathrm{i}\mathcal{Z}}+\mathrm{e}^{\mathrm{i}\mathcal{Z}})\right), \tag{3.12}$$

so that

$$q_{N,0}(\hat{z}) = \frac{15\pi^2}{4}\Lambda_0\ \cos\ \mathcal{Z}\ \mathrm{erfc}\left(\frac{\hat{z}}{\sqrt{\pi}}\right), q_{N,1}(\hat{z}) = -5\pi\mathrm{i}\Lambda_0\ \sin\ \mathcal{Z}\ \mathrm{erfc}\left(\frac{\hat{z}}{\sqrt{\pi}}\right) = -q_{N,-1}(\hat{z}), \tag{3.13}$$

where we have imposed the matching conditions $q_{N,k} \to 0$ as $\hat{z} \to +\infty$. Thus, from (3.11), we find that the leading-order expression for the remainder within the Stokes layer is given in terms of W by

$$W \sim \frac{5\pi\Lambda_0}{2}\varepsilon^{-4}\mathrm{e}^{-\frac{\pi}{2\varepsilon}}\left(4\ \sin\ \mathcal{Z}\ \sin\ \tilde{Z}+\frac{3\pi}{2}\cos\ \mathcal{Z}\right)\mathrm{erfc}\left(\frac{\hat{z}}{\sqrt{\pi}}\right). \tag{3.14}$$

3.3. SOLUTION OUTSIDE THE STOKES LAYER

As can be seen from the exponential decay of the right-hand side of (3.10) as $\hat{z} \to \pm\infty$, the remainder equation (3.4) is homogeneous to all relevant orders outside the Stokes layer, and the defining equation is then, to all desired orders,

$$R_{N_{\tilde{Z}\tilde{Z}\tilde{Z}}} + R_{N_{\tilde{Z}}} + \varepsilon(3R_{N_{\tilde{Z}\tilde{Z}z}} + R_{N_z} + 2U_0R_N) = 0. \tag{3.15}$$

Since U_0 is given in (2.10), standard multiple scales techniques lead to

$$R_N(z,\tilde{Z}) = a_1R_{N_1}(z,\tilde{Z}) + a_2R_{N_2}(z,\tilde{Z}) + a_3R_{N_3}(z,\tilde{Z}), \tag{3.16}$$

where a_1, a_2 and a_3 are arbitrary (exponentially small) constants and, to leading order as $\varepsilon \to 0$,

$$R_{N_1} \sim\ \mathrm{sech}\, z\ \cos\ \tilde{Z}, \tag{3.17}$$

$$R_{N_2} \sim -2\ \tanh\ z\ \mathrm{sech}\, z\ \sin\ \tilde{Z} + \mathrm{sech}^2 z, \tag{3.18}$$

$$R_{N_3} \sim 2(3\ \mathrm{sech}\, z - \cosh\ z - 3z\ \tanh\ z\ \mathrm{sech}\, z)\sin\ \tilde{Z} + 2\sinh\ z\ \cosh\ z + 3\tanh\ z + 3z\ \mathrm{sech}^2 z. \tag{3.19}$$

Matching (3.14) as $\hat{z} \to -\infty$ with (3.16) as $z \to 0^-$ gives for $z < 0$,

$$a_2 \sim \frac{15\pi^2}{2}\Lambda_0\varepsilon^{-4}\mathrm{e}^{-\frac{\pi}{2\varepsilon}}\cos\ \mathcal{Z}, a_3 \sim 5\pi\Lambda_0\varepsilon^{-4}\mathrm{e}^{-\frac{\pi}{2\varepsilon}}\sin\ \mathcal{Z}, \tag{3.20}$$

with a_1 negligible in comparison. An exponentially small term

$$5\pi\Lambda_0\varepsilon^{-4}\mathrm{e}^{-\frac{\pi}{2\varepsilon}}\left(\frac{3\pi}{2}\ \cos\ \mathcal{Z}R_{N_2}(z,\tilde{Z}) +\ \sin\ \mathcal{Z}R_{N_3}(z,\tilde{Z})\right) \tag{3.21}$$

is thus switched on as the Stokes line is crossed from positive to negative values of z.

We now note that R_{N_1} and R_{N_2} are constant multiples of the derivatives of U_0 with respect to $\tilde{Z}$ and z, respectively, as is to be expected in view of the translational invariance of (2.1). The terms R_{N_1} and R_{N_2} therefore represent translations in the origin of the independent variables $\tilde{Z}$ and z, respectively. The most important term that may get switched on is thus R_{N_3}, which contains terms which grow exponentially in z (see (3.19)). Only for the odd solutions described in Section 2.1, for which $\mathcal{Z} = 0$ or π, do we have $a_3 = 0$, indicating that *bona fide* oscillatory shock solutions occur in these cases only.

4. Effect of the Stokes switching on possible solutions

4.1. Oscillatory shock solutions: secondary shocks

Our aim in this section is to give an asymptotic description of solutions containing alternating sequences of local oscillatory and monotonic shocks, such as that illustrated in the numerical solution of Figure 5. This will enable us to give a rather complete analysis of the initial value problem in which (1.1) is subject to (1.3) or (1.4), with $0 < \varepsilon \ll 1$. We start by describing the behaviour in the local monotonic shock which follows an oscillatory shock.

We saw in Section 3 that an exponentially small multiple of R_{N_3} is generally switched on across the Stokes line. For $z < 0$ we can write the solution as

$$U(z, Z) \sim U_0(z, Z) + \cdots + \varepsilon^N U_N(z, Z) + \rho(\varepsilon) R_{N_3}(z, Z), \tag{4.1}$$

where U_0 is given in (2.10) and ρ satisfies

$$\rho \sim 5\pi \Lambda_0 \varepsilon^{-4} \mathrm{e}^{-\frac{\pi}{2\varepsilon}} \sin \mathcal{Z} \quad \text{a} \quad \varepsilon \to 0; \tag{4.2}$$

$\mathcal{Z}$ is related to ϕ by (2.16). We now aim to examine the effect that altering $\mathcal{Z}$, and hence ρ, has on the solution for negative values of z. We shall see that this can lead, for example, to the generation of a local monotonic shock solution, spaced a distance $O(1/\varepsilon)$ from the oscillatory shock (*cf.* Figure 5).

For z large and negative, we see that from (2.10), (3.19) and (4.1) that we may write

$$U(z, Z) \sim 1 - 4\mathrm{e}^{z} \sin(Z - \mathcal{Z}) + \cdots - \frac{1}{2}\rho \mathrm{e}^{-2z}. \tag{4.3}$$

The exponentially growing term switched on across the Stokes line thus ultimately contributes at leading order; this occurs for $z_2 = O(1)$ (corresponding to an anti-Stokes line), where

$$z = z_2 + \frac{1}{2}\log|\rho| - \log 2 = z_2 - \frac{\pi}{4\varepsilon} + 2\log\left(\frac{1}{\varepsilon}\right) + O(1). \tag{4.4}$$

In view of (4.3), for $z_2 = O(1)$ we can write

$$U(z, Z) \sim \hat{U}(z_2; \varepsilon) + |\rho|^{\frac{1}{2}} S(z_2, Z). \tag{4.5}$$

Here the second term is exponentially small and $\hat{U}$ is the solution to (1.1) (with z replaced by z_2) which satisfies

$$\hat{U}(z_2) \sim 1 - 2\mathrm{sgn}(\rho)\mathrm{e}^{-2z_2} \quad \text{as} \quad z_2 \to +\infty \tag{4.6}$$

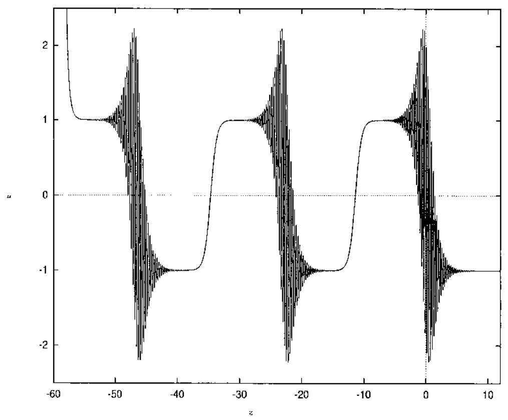

Figure 5. 'Multiple shock' solution obtained numerically for $\varepsilon = 0{\cdot}045$, $M = -4$, $\phi = 4{\cdot}0$.

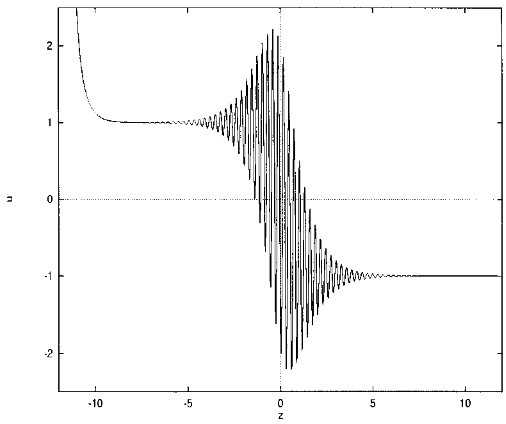

Figure 6. Numerical solution showing blow-up after one local oscillatory shock; $\varepsilon = 0{\cdot}045$, $M = -4$, $\phi = 1{\cdot}0$.

(*cf.* (1.4) with $M = -2\,\mathrm{sgn}(\rho)$). From (4.2) we have $\rho > 0$ for $\mathcal{Z} \in (0, \pi)$ and $\rho < 0$ for $\mathcal{Z} \in (\pi, 2\pi)$; for $\rho > 0$, the condition (4.6) is identical to that applicable in the case of monotonic shock solutions. Writing $\hat{U} \sim \hat{U}_0(z_2)$ as $\varepsilon \to 0$, we have

$$\frac{\mathrm{d}\hat{U}_0}{\mathrm{d}z_2} = 1 - \hat{U}_0^2. \tag{4.7}$$

Given (4.6), this has the solution

$$\hat{U}_0 = \tanh z_2, \tag{4.8}$$

for $\rho > 0$ and

$$\hat{U}_0 = \coth z_2 \tag{4.9}$$

for $\rho < 0$, which is singular at $z_2 = 0$, *i.e.*, at

$$z = -\frac{\pi}{4\varepsilon} + 2\log\left(\frac{1}{\varepsilon}\right) + O(1); \tag{4.10}$$

the solution blows up near this value of z, a further rescaling $z_2 = \varepsilon\xi, \hat{U} = \hat{\Phi}/\varepsilon$ being required to describe the local behaviour. This gives, to leading order,

$$\frac{\mathrm{d}^3\hat{\Phi}_0}{\mathrm{d}\xi^3} + \frac{\mathrm{d}\hat{\Phi}_0}{\mathrm{d}\xi} = -\hat{\Phi}_0^2, \tag{4.11}$$

$$\hat{\Phi}_0 \sim \frac{1}{\xi} \text{ as } \xi \to +\infty, \tag{4.12}$$

where we have matched with (4.9) to obtain (4.11). This determines $\hat{\Phi}_0$ only up to a translation of ξ; specifying it completely requires knowledge of $\hat{U}_1(z_2)$. The solution to (4.11) blows up according to

$$\hat{\Phi}_0 \sim \frac{60}{(\xi - \xi_0)^3} \text{ as } \xi \to \xi_0^+ \tag{4.13}$$

for some ξ_0; we believe this to be generic for real solutions to (1.1). This type of behaviour for $\rho < 0$ has been confirmed by closer inspection of numerical solutions, such as that shown in Figure 6.

For $\rho > 0$, the solution (4.8) is well-behaved for $z_2 = O(1)$, though a further Stokes line occurs in the neighbourhood of $z_2 = 0$ (*cf.* [12]), generated by the singularities in (4.8) at $z_2 = \pm i\pi/2$. It has been shown in [16, Chapter 4], [12] that on crossing the Stokes line at $z_2 = 0$, going from positive to negative values of z_2, an additional exponentially small (but exponentially growing) term

$$\frac{2\pi L}{\varepsilon^2} e^{-\frac{\pi}{2\varepsilon}} \cosh z_2 \cos Z_2 \tag{4.14}$$

is switched on in $\hat{U}(z_2, Z)$, where $L \approx 1{\cdot}25$ and $Z = Z_2 + \frac{\gamma}{2\varepsilon} \log(|\rho|/4)$, *i.e.*, $Z_2 = z_2/\varepsilon$. We can therefore write

$$\hat{U} \sim \tanh z_2 + \cdots + H(-z_2)\frac{2\pi L}{\varepsilon^2} e^{-\frac{\pi}{2\varepsilon}} \cosh z_2 \cos Z_2 \quad \text{as } \varepsilon \to 0, \tag{4.15}$$

where H is the Heaviside unit step function.

In the case $\rho > 0$, we must now consider the exponentially small oscillatory term S in (4.5), which is generated by the second term in (4.3). Writing $S \sim S_0(z_2, Z) + \varepsilon S_1(z_2, Z)$, we can obtain S_0 by a multiple-scales calculation on the homogeneous remainder equation (*cf.* (3.15)).

$$S_{ZZZ} + S_Z + \varepsilon(3S_{ZZz_2} + S_{z_2} + 2\hat{U}_0 S) = 0, \tag{4.16}$$

where $\hat{U}_0 = \tanh z_2$ (from (4.8)), given that

$$S_0(z_2, Z) \sim -2e^{z_2} \sin(Z - \mathcal{Z}) \quad \text{as } z_2 \to +\infty. \tag{4.17}$$

This yields (*cf.* [12])

$$S_0 = -4 \cosh z_2 \sin(Z_2 + \theta) \tag{4.18}$$

where, from (4.4), $\theta = \frac{1}{\varepsilon}(\frac{1}{2}\log|\rho| - \log 2) - \mathcal{Z}$ (modulo 2π). We can now write the solution for u in the case $z_2 = O(1)$, $\rho > 0$
in the form

$$U(z_2, Z) \sim \tanh z_2 - 4|\rho|^{\frac{1}{2}} \cosh z_2 \sin(Z_2 + \theta) + H(-z_2)\frac{2\pi L}{\varepsilon^2} e^{-\frac{\pi}{2\varepsilon}} \cosh z_2 \cos Z_2, \tag{4.19}$$

as $\varepsilon \to 0$, having used (4.5), (4.15) and (4.18) to obtain this result. Note that in (4.19) the algebraic expansion involving $\hat{U}_1$ etc. has been omitted, so the expression is not an asymptotic one in the usual sense. It does, however, contain all the terms in the asymptotic expansion that we need, namely the leading-order solution and the dominant exponentially growing terms; we write expression of this type on a number of occasions.

Hence, the solution for $z < 0$ differs in form depending on the sign of ρ, which from (4.2) and (2.44) is seen to be that of $\sin \mathcal{Z}$. The case $\rho < 0$ leads to blow-up and $\rho > 0$ to a monotonic shock-type solution, with further implications to be discussed in Section 4.2.

The case $\rho = 0$ corresponds to the oscillatory shock solution (consistent with the values of $\mathcal{Z} = 0, \pi$ obtained in Section 2.1 on imposing the constraint that the solution be odd).

We now extend our approach to the whole of the real z-axis, and show that the solution typically alternates between local oscillatory and monotonic shocks as z decreases, unless and until the solution blows up, which can only occur after a local oscillatory shock.

4.2. Oscillatory shock solutions: multiple shocks

4.2.1. *Derivation of difference equations*

We have seen how, as z decreases along the real axis, it is possible to obtain a local oscillatory shock followed after a distance $z = O(1/\varepsilon)$ by a local monotonic shock. We now extend the analysis to encompass multiple local shocks of alternating type.

We start by summarising the 'input-output' relationships for a local shock:

(i) Local oscillatory shock (see Section 3).

Suppose we choose the origin of z such that we have the asymptotic representation

$$U(z, Z) \sim -1 + M^{(1)}(\varepsilon)e^{-z} \sin(Z + \phi) + \mathcal{A}(\varepsilon)e^{2z} \quad \text{as } z \to \infty, \varepsilon \to 0, \tag{4.20}$$

where $M^{(1)}(\varepsilon)$ is the exact value of M in (1.3) corresponding to the odd oscillatory shock with $\phi = \phi^{(1)}(\varepsilon)$ (the expansions for which are given by (2.17)–(2.18), and where $\mathcal{A}(\varepsilon)$ is exponentially small). Then, using (3.19), we have

$$U(z, Z) \sim \hat{U}(z, Z) + 2\mathcal{A}(\varepsilon)R_{N_3}(z, Z) \quad \text{for } z = O(1) \quad \text{with } z > 0, \tag{4.21}$$

where $\hat{U}(z, Z)$ is the solution to the initial-value problem in which (1.1) is subject to

$$\hat{U}(z, Z) \sim -1 + M^{(1)}(\varepsilon)e^{-z} \ \sin(Z + \phi) \quad \text{as } z \to +\infty. \tag{4.22}$$

The leading-order solution for $\hat{U}$ is thus of the form given in (2.10) and a quantity

$$\rho(\varepsilon)R_{N_3} \tag{4.23}$$

is switched on as the Stokes line at $z = 0$ is crossed, where from (4.2),

$$\rho(\varepsilon) \sim -5\pi\Lambda_0\varepsilon^{-4}e^{-\frac{\pi}{2\varepsilon}} \sin(\phi - \phi^{(1)}), \tag{4.24}$$

ρ being zero to all orders when $\phi = \phi^{(1)}$ or $\phi = \phi^{(2)}$. Hence, in view of (3.19),

$$U(z, Z) \sim 1 + M^{(1)}(\varepsilon)e^{z} \sin(Z + \phi - 2\phi^{(1)}) - \mathcal{C}(\varepsilon)e^{-2z} \quad \text{as } z \to -\infty, \varepsilon \to 0, \tag{4.25}$$

where the exponentially small quantity $\mathcal{C}$ is given by

$$\mathcal{C}(\varepsilon) \sim \mathcal{A}(\varepsilon) + \rho(\varepsilon)/2. \tag{4.26}$$

If $\mathcal{C} > 0$ a local monotonic shock is then generated as z decreases, whereas if $\mathcal{C} < 0$ blow-up occurs, in each case near $z = -\frac{1}{2}\log(2/|\mathcal{C}|)$. It should be emphasised that (4.20) and (4.25) (and (4.27) and (4.28) below) are matching conditions – they do not describe the behaviour of an exact solution as $|z| \to \infty$, but rather the outer limits of the $z = O(1)$ asymptotic solutions.

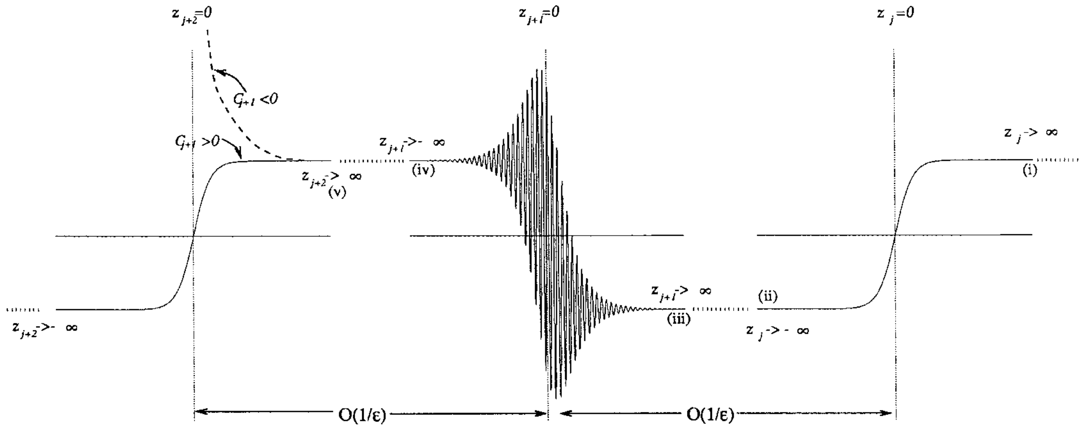

Figure 7. Schematic of the solution u against z in three different (two monotonic and one oscillatory) local shock regions. The relationships between the coefficients in neighbouring shocks are given by (i) (4.27); (ii) (4.28); (iii) (4.20); (iv) (4.25); (v) (4.27).

(ii) Local monotonic shock (see (4.8)–(4.19)). If

$$U(z, Z) \sim 1 - 2\mathrm{e}^{-2z} + \mathcal{B}(\varepsilon)\mathrm{e}^{z} \sin(Z + \theta) \quad \text{as } z \to +\infty, \varepsilon \to 0, \tag{4.27}$$

where $\mathcal{B}(\varepsilon)$ is an exponentially small constant, then we have the matching condition

$$U(z, Z) \sim -1 + 2\mathrm{e}^{2z} + \mathcal{D}(\varepsilon)\mathrm{e}^{-z} \sin(Z + \psi) \quad \text{as } z \to -\infty, \varepsilon \to 0 \tag{4.28}$$

where $\mathcal{D} \geq 0$ with

$$\mathcal{D} \sim (P^2 + \mathcal{B}^2 + 2P\mathcal{B} \sin\theta)^{\frac{1}{2}}, \mathcal{D} \sin\psi \sim P + \mathcal{B} \sin\theta, \mathcal{D} \cos\psi \sim \mathcal{B} \cos\theta, P = \pi L\varepsilon^{-2}\mathrm{e}^{-\frac{\pi}{2\varepsilon}}. \tag{4.29}$$

Thus if $\mathcal{D} \neq 0$ (4.28) generates a local oscillatory shock about $z = -\log(-M^{(1)}/\mathcal{D})$.

We are now in a position to describe how an alternating sequence of local monotonic and oscillatory shocks is generated. This is illustrated in Figure 7 and we introduce a local variable z_j and a subscript j on the constants $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, θ, ϕ and ψ (essentially as introduced above) when they occur in the jth local shock. We take $z_1 \equiv z$.

Thus if the jth local shock is a monotonic shock, we have from (4.29)

$$\mathcal{D}_j \sin\psi_j \sim P + \mathcal{B}_j \sin\theta_j, \quad \mathcal{D}_j \cos\psi_j \sim \mathcal{B}_j \cos\theta_j, \tag{4.30}$$

where

$$P = \pi L\varepsilon^{-2}\mathrm{e}^{-\frac{\pi}{2\varepsilon}}. \tag{4.31}$$

The $(j+1)$th shock is an oscillatory shock located about $z_{j+1} = 0$, where (in view of (2.18)) we take

$$z_{j+1} \sim z_j + \log(4/\mathcal{D}_j), \tag{4.32}$$

and from (4.28) and (4.20) (with z replaced by z_{j+1} and $\phi - \phi^{(1)}$ by ϕ_{j+1}) we have that

$$\mathcal{A}_{j+1} = \mathcal{D}_j^2/8, \tag{4.33}$$

$$\phi_{j+1} = -\frac{1}{\varepsilon}\log(4/\mathcal{D}_j) + \psi_j + \pi \quad \text{(modulo } 2\pi\text{)}, \tag{4.34}$$

and, from (4.26) and (4.2),

$$\mathcal{C}_{j+1} \sim \mathcal{A}_{j+1} - Q\,\sin\phi_{j+1}, \tag{4.35}$$

where

$$Q = \frac{5\pi}{2}\Lambda_0\varepsilon^{-4}\mathrm{e}^{-\frac{\pi}{2\varepsilon}}. \tag{4.36}$$

If $\mathcal{C}_{j+1} > 0$ (*cf.* (4.6)–(4.8)), this then generates a further local (monotonic) shock located about $z_{j+2} = 0$, with

$$z_{j+2} \sim z_{j+1} + \frac{1}{2}\log(2/\mathcal{C}_{j+1}), \tag{4.37}$$

$$\mathcal{B}_{j+2} \sim -2\sqrt{2\mathcal{C}_{j+1}}, \tag{4.38}$$

$$\theta_{j+2} \sim -\frac{1}{2\varepsilon}\log(2/\mathcal{C}_{j+1}) + \phi_{j+1} \quad \text{(modulo } 2\pi\text{)}; \tag{4.39}$$

note in (4.32), (4.37) that j and $j+2$ necessarily denote monotonic shocks and $j+1$ an oscillatory one. (4.30)–(4.39) provide a coupled system of difference equations relating, say, $\mathcal{B}_{j+2}$ and θ_{j+2} to $\mathcal{B}_j$ and θ_j.

We note that this system breaks down if $\mathcal{C}_{j+1}$ is negative, in which case the solution blows up (as described earlier), or if either $\mathcal{C}_{j+1}$ or $\mathcal{D}_j$ are zero, in which case the state $u \to 1$ or $u \to -1$, respectively, persists as $z \to -\infty$.

Some comments regarding the above system are needed relating to the explicit appearance of the small parameter ε in the asymptotic expressions. Firstly, the appearance of terms in $\log(\mathcal{D}_j)/\varepsilon$ and $\log(\mathcal{C}_{j+1})/\varepsilon$ requires that $\mathcal{D}_j$ and $\mathcal{C}_{j+1}$ be calculated at each stage correctly at $O(\varepsilon)$ and it is not immediately clear that all the necessary ($O(\varepsilon)$) correction terms have been included in (4.30)–(4.39); in any case, the above formulation is expected at the very least to capture the appropriate qualitative behaviour. Secondly, not all of the terms are of the same order in ε and we next describe the main distinguished limit. However, which terms enter at leading order depends on the shock being described and for the discussion of the next subsection all the terms are relevant at some point; (4.30)–(4.39) can thus be viewed as providing a convenient uniformly valid approximation. The distinguished limit we now describe has $\mathcal{A}_{j+1}$, $\mathcal{B}_j^2$, $\mathcal{C}_{j+1}$ and $\mathcal{D}_j^2$ all of $O(Q)$ (*cf.* (4.35)) so the P terms in (4.30) are negligible, yielding (since $\mathcal{D}_j > 0$, $\mathcal{B}_j < 0$)

$$\mathcal{B}_j \sim -\mathcal{D}_j, \quad \psi_j \sim \pi + \theta_j,$$

and (4.33), (4.38) then give

$$\mathcal{A}_{j+1} \sim \mathcal{C}_{j-1};$$

(4.34), (4.35) and (4.39) thus reduce to

$$\mathcal{C}_{j+1} \sim \mathcal{C}_{j-1} - Q\,\sin\phi_{j+1}, \tag{4.40}$$

$$\phi_{j+1} \sim \phi_{j-1} - \frac{1}{\varepsilon}\log(2/\mathcal{C}_{j-1}) \quad (\text{modulo } 2\pi). \tag{4.41}$$

The final term in (4.41) is of $O(1)$, despite the $1/\varepsilon$ prefactor and the exponential smallness of $\mathcal{C}_{j-1}$, because it is to be taken modulo 2π; the small parameter in (4.41) cannot thus be readily exploited, clarifying the sensitive dependence of the solution on ε. However, under the scaling

$$\mathcal{C}_j \sim Q\left(\frac{c}{\varepsilon} + c_j + d\right),$$

where the positive constant c is independent of ε (with the $\mathcal{C}_j$ thus being almost equal) and $c_j, d = O(1)$, the dependence on ε can be eliminated from the leading-order problem which, defining the constant d by

$$\frac{\mathrm{d}}{c} = \frac{1}{\varepsilon}\log\left(\frac{2\varepsilon}{Qc}\right) \quad (\text{modulo } 2\pi),$$

takes the form

$$c_{j+1} \sim c_{j-1} - \sin\phi_{j+1}, \quad \phi_{j+1} \sim \phi_{j-1} - c_{j-1}/c,$$

equivalent to the standard map, which seems to be generic in this context and is itself well-known to exhibit extremely delicate behaviour.

4.2.2. *Solutions arising from given initial conditions*
We now note the form of the solutions which may arise for given initial conditions (1.3) or (1.4). Of particular interest will be the existence of regular shocks, solitary waves and periodic solutions. The first two of these occur when either $\mathcal{C}_{j+1} = 0$ or $\mathcal{D}_j = 0$.

We first observe that imposing (1.4) with $M = -2$ (we are free to choose this value by translating z if $M < 0$; if $M > 0$ then the solution blows up (*cf.* (4.6)–(4.10))), then in (4.27) $\mathcal{B}_1 = 0$, and hence from (4.30), $\mathcal{D} > 0$, implying the non-existence of a true monotonic shock, consistent with earlier results (*e.g.* [6]).

Similarly, our previous results concerning the existence of oscillatory shocks follow in that if we impose (1.3) with, by translating z, $M_0 = M^{(1)}$, then we have $\mathcal{A}_1 = 0$ in (4.20) and equation (4.35) implies that $\mathcal{C}_1 = 0$ for $\phi = \phi^{(1)}$ and $\phi = \phi^{(2)}$.

We now investigate what may occur after two or more local shocks.
Homoclinic connections We shall consider homoclinic connections (solitary wave solutions) with $u \to 1$ as $|z| \to \infty$; those with $u \to -1$ as $|z| \to \infty$ can be obtained from these via $u \mapsto -u, z \mapsto -z$.

Imposing (1.4) (with $M = -2$), we have for the first local shock, from (4.27) that $\mathcal{B}_1 = 0$ and therefore from (4.30)

$$\mathcal{D}_1 \sim P, \tag{4.42}$$

$$\psi_1 \sim \frac{\pi}{2}. \tag{4.43}$$

This then generates a second local (oscillatory) shock, where from (4.32)–(4.35)

$$z_2 \sim z_1 + \frac{\pi}{2\varepsilon} - 2\log\left(\frac{1}{\varepsilon}\right) + \log\frac{4\pi}{L}, \tag{4.44}$$

$$\mathcal{A}_2 \sim \frac{P^2}{8}, \tag{4.45}$$

$$\phi_2 \sim -\frac{1}{\varepsilon}\left(\frac{\pi}{2\varepsilon} - 2\,\log\left(\frac{1}{\varepsilon}\right) + \log\frac{4\pi}{L}\right) + \frac{3\pi}{2} \quad (\text{modulo } 2\pi), \tag{4.46}$$

$$\mathcal{C}_2 \sim \frac{P^2}{8} - Q\,\sin\phi_2. \tag{4.47}$$

Thus for a '2 local shock' homoclinic connection (*e.g.*, Figure 1(c)), we require $\mathcal{C}_2 = 0$, hence from (4.46), (4.47)

$$\cos\left(\frac{1}{\varepsilon}\left(\frac{\pi}{2\varepsilon} - 2\,\log\left(\frac{1}{\varepsilon}\right) + \log\frac{4\pi}{L}\right)\right) \sim -\frac{\pi L}{20\Lambda_0}\mathrm{e}^{-\frac{\pi}{2\varepsilon}}. \tag{4.48}$$

Thus we have an implicit equation for ε to determine when such homoclinic connections are possible, and we therefore infer that such solutions exist only at discrete values of ε, with $\varepsilon \sim (2m)^{-\frac{1}{2}}$ for large integer m.

The distance between the centres of each local shock can be determined from (4.44) as

$$\frac{\pi}{2\varepsilon} - 2\,\log\left(\frac{1}{\varepsilon}\right) + \log\frac{4\pi}{L}. \tag{4.49}$$

We note from (4.47)–(4.48) that $\sin\phi_2 = O(\exp(-\pi/2\varepsilon))$ increases so the value of ϕ in the local oscillatory shock is exponentially close to that required for a true oscillatory shock ($\sin\phi = 0$). This is again in agreement with the observations of Yang [17] and emphasises the exponential sensitivity of the problem for small ε.

For a '4 local shock' homoclinic connection (see Figure 8), we require that $\mathcal{C}_4 = 0$, where the constants for the first two local shocks are given by (4.42)–(4.47). We can impose $\mathcal{C}_4 = 0$ and work backwards to give, from (4.35)

$$\mathcal{A}_4 \sim Q\,\sin\phi_4, \tag{4.50}$$

thus from (4.33) and (4.50)

$$\mathcal{D}_3 \sim (8Q\,\sin\phi_4)^{\frac{1}{2}}, \tag{4.51}$$

and from (4.34)

$$\psi_3 \sim \frac{\pi}{4\varepsilon^2} - \frac{2}{\varepsilon}\log\frac{1}{\varepsilon} - \frac{1}{2\varepsilon}\log\left(\frac{5\pi}{4}\Lambda_0\,\sin\phi_4\right) + \phi_4 + \pi \quad (\text{modulo } 2\pi). \tag{4.52}$$

From (4.30), since $\mathcal{D}_3 \gg P$, we have that

$$\mathcal{B}_3 \sim -(8Q\,\sin\phi_4)^{\frac{1}{2}}, \tag{4.53}$$

$$\theta_3 \sim \psi_3 + \pi \sim \frac{\pi}{4\varepsilon^2} - \frac{2}{\varepsilon}\log\frac{1}{\varepsilon} - \frac{1}{2\varepsilon}\log\left(\frac{5\pi}{4}\Lambda_0\,\sin\phi_4\right) + \phi_4 \quad (\text{modulo } 2\pi), \tag{4.54}$$

therefore for the '4 shock' homoclinic connection, we need (4.46), (4.47), (4.53) and (4.54) to satisfy (4.38) and (4.39). These transcendental equations can be solved numerically, giving discrete values of ε and θ_4 and we thus again expect to see such solutions only for discrete values of ε, subject to the additional constraints that $\mathcal{C}_2, \mathcal{A}_4 > 0$ in order that the solution does not blow-up. We note from (4.37) and (4.47) and from (4.32) and (4.51) that the distances between the second and third and between the third and fourth local shocks, respectively, are both given by

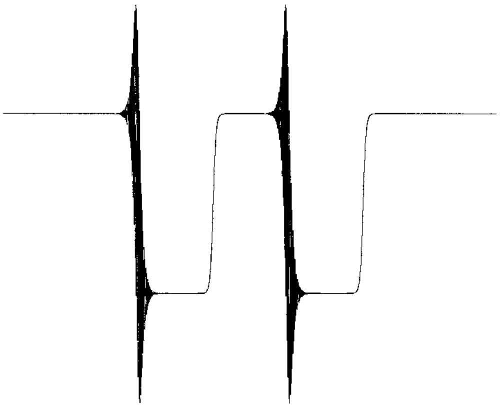

Figure 8. Schematic of '4 local shock' homoclinic connection.

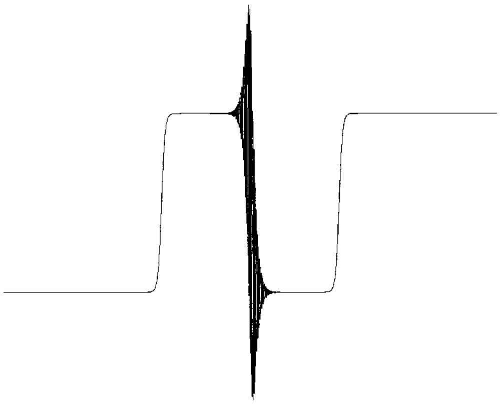

Figure 9. Schematic of '3 local shock' heteroclinic connection (regular shock).

$$\frac{\pi}{4\varepsilon} - 2\,\log\left(\frac{1}{\varepsilon}\right) + O(1). \tag{4.55}$$

Note that this distance is approximately half of that between the first and second local shocks (4.49). Confirmation of these distances is provided by numerical solutions (see Section 5).

The calculations above can be generalised in an obvious way to homoclinic connections containing $2m$ local shocks, where m is an integer.

Heteroclinic connections The existence of single oscillatory shock solutions and non-existence of single monotonic regular shock solutions having been established, we now consider odd solutions containing three local shocks. First, we consider one of the type shown in Figure 9. Such odd solutions necessitate that $\sin\theta_2 = 0$, *i.e.*, nothing is switched on across a Stokes line in the second local shock. Having already obtained the value of θ_2 in (4.46), we can immediately write down the required asymptotic condition as

$$\cos\left(\frac{1}{\varepsilon}\left(\frac{\pi}{2\varepsilon} - 2\,\log\left(\frac{1}{\varepsilon}\right) + \log\frac{4\pi}{L}\right)\right) = 0. \tag{4.56}$$

Thus again, we can obtain solutions to this equation, and therefore give an asymptotic construction of the 3-shock solution, only for discrete values of ε.

For the 3-shock case of the type shown in Figure 10, odd solutions require that the second shock be odd in z_2. From (4.27) and (4.28), we thus have that $\mathcal{B}_2 \sim -\mathcal{D}_2$ and $\theta_2 \sim -\psi_2 + \pi$ and therefore from (4.30)

$$\sin\theta_2 \sim -\frac{P}{2\mathcal{B}_2}. \tag{4.57}$$

Applying the oscillatory shock initial conditions (1.3), we have $\mathcal{A}_1 = 0$, ϕ_1 arbitrary, and can give expressions for $\mathcal{B}_2$ and θ_2 using symmetry arguments from (4.51) and (4.52) respectively, as

$$\mathcal{B}_2 \sim -(-8Q\,\sin\phi_1)^{\frac{1}{2}}, \tag{4.58}$$

$$\theta_2 \sim -\frac{\pi}{4\varepsilon^2} + \frac{2}{\varepsilon}\log\frac{1}{\varepsilon} + \frac{1}{2\varepsilon}\log\left(-\frac{5\pi}{4}\Lambda_0\,\sin\phi_1\right) + \phi_1 \quad (\text{modulo } 2\pi). \tag{4.59}$$

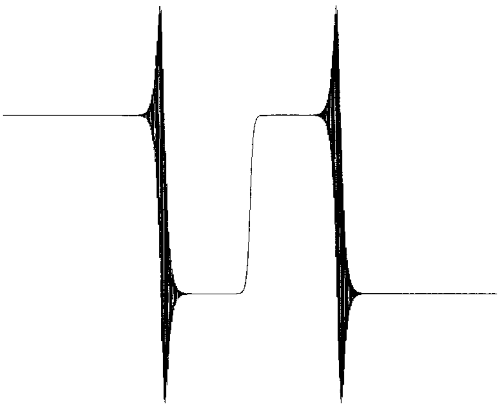

Figure 10. Schematic of a '3 local shock' heteroclinic connection (oscillatory shock).

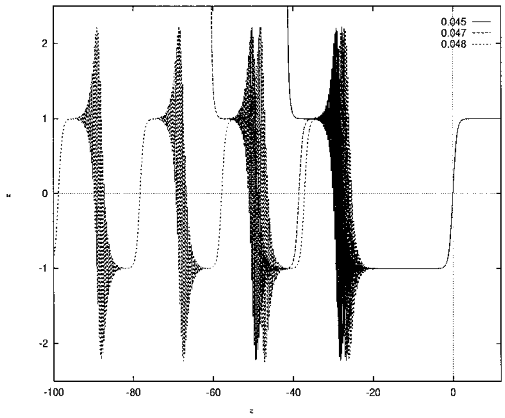

Figure 11. Solutions obtained numerically for (1.4) with $M = -2$ for $\varepsilon = 0{\cdot}045$, $\varepsilon = 0{\cdot}047$, $\varepsilon = 0{\cdot}048$

Thus, for any value of ε, we can substitute (4.58) and (4.59) in (4.57) and solve for ϕ_1; the left-hand side of (4.59) is typically exponentially small. Therefore, provided that $\mathcal{C}_1 > 0$, we can find odd 3-shock 'oscillatory' heteroclinic connections for any small value of ε, a conclusion which is consistent with boundary condition counting arguments.

Again, the above calculations can be extended to $2m + 1$ shock heteroclinic connections, where m is an integer.

To summarise, odd heteroclinic connections with $u \to 1$ as $z \to -\infty$ and $u \to -1$ as $z \to +\infty$ exist for any small ε containing (presumably) any $O(1)$ odd number of local shocks. Odd heteroclinic connections with $u \to -1$ as $z \to -\infty$ and $u \to 1$ as $z \to +\infty$ also exist at discrete values of ε containing three and (presumably) more local shocks; the corresponding solution containing only one local shock occurs only for $\varepsilon = 0$. Homoclinic connections with $u \to 1$ as $|z| \to \infty$ and $u \to 1$ as $z \to +\infty$ (or $u \to -1$ as $|z| \to \infty$) may exist containing any even number of local shocks, but will do so only at discrete values of ε. Such comments are consistent with, for example, the numerical observations of Hooper and Grimshaw [10]; see also the analysis of Yang [17].

5. Numerical results

The solution to (1.1) has been obtained numerically for the initial value problem (1.3). In most cases, $M = -4$ (*cf.* (2.16)) was used, though to obtain a more accurate value of the critical values $\phi^{(1)}$ and $\phi^{(2)}$ (see Figure 2) we included the correction term to M from (2.18). As can be seen from Figure 2, this successfully produced shocks which are odd about a value of z extremely close to zero. The NAG stiff-solver routine D02EBF was used throughout with double precision arithmetic. The value of ϕ was varied, with initial conditions for u, u' and u'' being evaluated from (1.3), and imposed at a large values of z.

It was seen in Section 4.1 that there existed two critical values, $\mathcal{Z} = 0$ and π, for which a true oscillatory shock solution occurs. For $\sin \mathcal{Z} > 0$, further local shocks are generated as z decreased, whereas for $\sin\ \mathcal{Z} < 0$, the solution blows up at some $z = -\pi/(4\varepsilon) +$

$2\log(1/\varepsilon) + O(1)$. Given the relationship between ϕ and $\mathcal{Z}$ (see (2.16) and (2.17)), we can deduce the following behaviour for $\phi \in [0, 2\pi)$:

$$\phi^{(2)} < \phi < \phi^{(1)} \Rightarrow \text{further local shocks}, \tag{5.1}$$

$$\phi < \phi^{(2)}, \phi > \phi^{(1)} \Rightarrow \text{blow-up}. \tag{5.2}$$

These are consistent with our numerical solutions, observed for various values of $\phi \in [0, 2\pi)$, which we illustrate for $\phi = 1(< \phi^{(2)})$, which leads to immediate blow-up (see Figure 6), and $\phi = 4(\in (\phi^{(2)}, \phi^{(1)})$, which generates further local shocks (see Figure 5).

Numerical values for $\phi^{(1)}$ and $\phi^{(2)}$ were obtained by starting from a value of ϕ that led to blow-up after one oscillatory shock and another that led to further shocks, then homing in on the value at which the transition between these two regimes occurs. For $\varepsilon = 0{\cdot}045$, we found that $\phi^{(1)} \approx 6{\cdot}145$ and $\phi^{(2)} \approx 3{\cdot}003$. The results of the numerical integration with these values of ϕ, which are in good agreement with the asymptotic values obtained from (2.17) (namely $\phi^{(1)} \approx 6{\cdot}141$, $\phi^{(2)} \approx 2{\cdot}99$), are given in Figure 2.

We also solved (1.1) numerically for the initial conditions (1.4) with (without loss of generality) $M = -2$. As has been discussed in Section 4, we expect to see a local monotonic shock followed by a local oscillatory shock, and either more local shocks or blow-up to then occur, depending sensitively on the value of ε. Examples of this are given in Figure 11 for three different values of ε.

The distances between successive local shocks are given asymptotically by (4.49), (4.55) and agree well with the numerical solutions shown in Figure 5 and Figure 11. We note that the distance between the first (from right to left) pair of local shocks in Figure 11 is given by (4.49), whereas all other distances are given by (4.55), and for $\varepsilon = 0{\cdot}045$ these are 31 and 11, respectively.

Detailed comparison of the numerical and asymptotic solutions is fraught with difficulties. Using Fortran double precision arithmetic implies that only quantities within fourteen significant figures of the leading-order solution have any chance of being captured by the numerics. For the oscillatory shock, for example, the terms switched on are of $O(\varepsilon^{-4}\exp(-\frac{\pi}{2\varepsilon}))$. For these to be captured numerically, we thus require, roughly speaking, that ε satisfy

$$\varepsilon^{-4}\mathrm{e}^{-\frac{\pi}{2\varepsilon}} \geq 10^{-13}, \tag{5.3}$$

which gives $\varepsilon \geq 0{\cdot}036$. For the monotonic shock solution, the term switched on is of $O(\varepsilon^{-2}\exp(-\frac{\pi}{2\varepsilon}))$, requiring $\varepsilon \geq 0{\cdot}043$. Hence, any value of ε much lower than $\varepsilon = 0{\cdot}045$, the smallest value used here, would presumably lead to the crucial Stokes switchings not being adequately captured by the numerical scheme. Such comments apply much more generally to problems involving asymptotics beyond all orders; numerical schemes may give qualitatively erroneous results due to rounding errors even for values of ε which are not particularly small.

6. Discussion

We have obtained asymptotic solutions to the Kuramoto-Sivashinsky equation for real z as $\varepsilon \to 0$, in both the oscillatory shock and monotonic shock cases, with particular reference to the effects that the exponential growth of exponentially small terms resulting from the Stokes phenomenon have far beyond the Stokes line; we have thus analysed such solutions for larger ranges of the independent variable than previously. We have seen that the result is an alternating sequence of local oscillatory and monotonic shocks, with the solution usually eventually blowing up, which is possible only after a local oscillatory shock; heteroclinic and homoclinic connections containing different numbers of local shocks can also be constructed, however. Although it is relatively easy to calculate oscillatory shocks numerically, it is very difficult to obtain connections with more than one local shock in this way (*cf.* Hooper and Grimshaw [10]; we note, however, that the bisection process used in the former case could again be applied). Good agreement between the asymptotic and numerical solutions has been demonstrated, although it should be stressed that any numerical scheme must accurately capture terms of sizes down to about $\exp(-\pi/2\varepsilon)$ to provide reliable results.

We have chosen in this paper to focus attention on a single ordinary differential equation in order to give a detailed description of the arguments underpinning the the optimal truncation approach to the analysis of beyond-all-orders effects. In particular, we have carefully derived, and given numerical validation for, the predominance of the Fourier components $k = 0, \pm 1, \pm 2$ in the large n behaviour (2.47). It should be stressed that the same techniques apply directly to much more general classes of equation (*cf.* [12]) and we hope that the insight provided by the detailed analysis of the example above will assist in pursuing more concise treatments of beyond-all-orders effects in other multiple-scales problems, with the factorial over power *ansatz* (*cf.* (2.47)) as usual playing a crucial role.

The current analysis provides a further example of the role of exponentially small terms in selecting a discrete set of solutions from a family all of whose members appear to provide valid solutions at each order, namely the selection of two oscillatory shocks (both of which are odd) from a plausible one-parameter family. In Section 4 we have attempted to describe how broader classes of solution can be analysed, with the exponential sensitivity of the problem (with ultimately very different solutions resulting from initial data which are exponentially close) being illustrated; while this sensitivity militates against giving an exhaustive account of the multiple-shock solutions here, the difference equation formulation of Section 4.2.1 represents a substantial simplification of the original system.

The linearisation at infinity implies that the current problem contains precisely two scales, z and Z, and we have attempted to exploit such information to the full. The beyond-all-orders analysis furnishes some additional insight into the multiple-scales method; in particular, the usual procedure of treating the two scales z and Z as being distinct independent variables (despite their being related via $Z = \gamma z/\varepsilon$) is acceptable to all orders but their interrelation proves crucial when performing the Stokes line analysis. Thus, in going from (3.6) to (3.7) z/ε has been rewritten in terms of Z and this is the only stage in the analysis of Sections 2 and 3 in which the two variables are not treated as independent (other than the separate discussion of the boundary value problem involving (1.7)); it is, moreover, a key step in accounting correctly for exponentially small terms. A closely related point is that in going from (1.1) to the multiple-scales form (2.1) we apparently gain a second translation invariant (in Z as well as in z), which is responsible for the existence to all orders of the family (2.21) with arbitrary $\mathcal{Z}$. This also clarifies why the relation between z and Z must necessarily play a role in accounting for exponentially-small terms. Finally, the relationship between the two variables is also important in Section 4 in relating translations of z between neighbouring shocks to those in Z. The Stokes smoothing in these multiple-scales problems is, as in earlier studies not involving multiple-scales effects, via an error function.

We conclude by noting that, for reasons outlined above, purely numerical techniques are impractical for problems of this ilk, even for values of ε which would not usually be regarded as particularly small. Exponentially accurately asymptotics thus has an essential role to play, in combination with numerics and rigorous analysis, in the treatment of a very wide class of such problems.

Appendix A, Other solutions to (2.7)–(2.9)

In seeking solutions to (1.1), other than those satisfying (1.3), we return to (2.7)–(2.9); see also [3]. Without loss of generality, we can choose $B_0(z) \equiv 0$. Integrating (2.7), (2.9) gives the relationship

$$C_0^2 = \frac{\mu}{A_0^2} + 1 - \frac{1}{4}A_0^2 \tag{A.1}$$

for $A_0 \neq 0$ ($A_0 = 0$ gives the leading order monotonic shock solution $u = \tanh\, \tilde{z}$). The centre (given by $\mu = -1$) has the solution

$$U_0 = \sqrt{2}\, \cos\, \tilde{Z}, \tag{A.2}$$

and the heteroclinic orbit ($\mu = 0$) for which $A_0 \neq 0$ represents the oscillatory shock solution.

Appendix B, Justification for neglecting $\mathbf{B_{n,k}}$ terms for which $|\mathbf{k}| > 2$

The purpose of this appendix is to give further justification for OB(5). Substituting (2.31) into (2.29), it can be seen that, of the nonlinear terms, only those for which $j = 0$ or $j = n$ are relevant at leading order.
Thus we have for $k > 0$

$$\begin{aligned} -k(k^2-1)\Gamma(n+\alpha_k+2)\Lambda_k \sim -(3k^2-1)(n+1)\Gamma(n+\alpha_k+1)\Lambda_k - n(n^2-1)\Gamma(n+\alpha_k-1)\Lambda_k \\ +3kn(n+1)\Gamma(n+\alpha_k)\Lambda_k - 2\Gamma(n+\alpha_{k-1}+1)\Lambda_{k-1} \\ +2\Gamma(n+\alpha_{k+1}+1)\Lambda_{k+1}. \end{aligned} \tag{B.1}$$

Equating the possible leading order terms in n gives

$$-nk(k-1)(k-2)\Lambda_k \sim -2n^{\alpha_{k-1}-\alpha_k}\Lambda_{k-1} + 2n^{\alpha_{k+1}-\alpha_k}\Lambda_{k+1}, \tag{B.2}$$

implying that

$$\alpha_{k-1} - \alpha_k = 1 \quad \text{for}\, k \geq 3, \tag{B.3}$$

so that

$$(k^3 - 3k^2 + 2k)\Lambda_k = 2\Lambda_{k-1}. \tag{B.4}$$

Thus, from (2.31) and (B.3), we have that

$$O(B_{n,k}) = O\left(\frac{1}{n}B_{n,k,1}\right) \tag{B.5}$$

for $k \geq 3$. The equivalent result for $k \leq -3$ follows immediately from OB(1).

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